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Optimal Sensor Locations for System Identification

March 1988

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FOREWORD

This final report was submitted by the Southeastern Center for Electrical Engineering Education, St. Cloud, Fl, in support of contract F04611-84-C-0028 with the Air Force Astronautics Laboratory (AFAL), Edwards Air Force Base, CA. AFAL Project Manager was Alok Das.

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This report deals with four major topics which impact, in a substantive manner, our ability to identify large, spatially extended structural systems in space with modest resources.

A) The first topic studies the uniqueness in the identification of a cantilever, boom-type structural system which is modelled by finite elements and is subjected to a base motion. It is shown that the entire stiffness distribution of the structure can be determined uniquely if the rotational and displacement time histories of response are available at the base node and the node adjacent to the base.

B) The second topic deals with the development of a methodology for optimally placing sensors in a large spatially extended structural system so that data collected from those locations would yield maximum information about the parameters to be identified. A preliminary methodology applicable to systems that can be adequately described by a set of ordinary differential equations has been developed and validated. It is shown that the judicious

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tradeoffs between identification and control; uniqueness of identification; multidegree-of-freedom systems.

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placement of sensors can not only reduce the costs involved in the collection, storage and processing of data with regard to identification of the necessary parameters, but that proper sensor placement can affect the accuracy of the identified results in a dramatic manner.

C) The third topic deals more directly with experimental design. It aims at tailoring the measurement stream to minimize data handling and processing related to system identification. It debunks the myth that the larger the data set one collects on the response of a dynamic system, the better off one is in identifying its parameters. It shows that certain data points may have no bearing on improving the accuracy of the parameters which are to be identified and therefore need not be recorded, stored or processed. A deeper understanding of these results will have a significant effect on in-flight identification of large space structures.

D) The fourth topic deals with an analysis of the tradeoffs between control and identification. The duality of the concepts of identification and control are studied in a quantitative manner and an illustrative example is shown amplifying the analytical results.

The report has several new and unexpected results, some of which may appear non-intuitive because of the nature of the inverse problem of identification which it deals with.

ACKNOWLEDGEMENTS

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Several people were *instrumental in generating* the interesting results presented here and I am thankful to all of them. Dr. Alok Das provided, all along the course of this work, very helpful comments, valuable suggestions, constructive criticism and prompt feedback. I especially thank him for the long hours of discussions which he engaged in with me and for his instant enthusiasm and support. Mr. L. K. Slimak discussed the work from time to time and provided several insightful comments and practical suggestions. Major David J. Olkowski and Lt. Eric Dale also participated in the discussions and contributed significantly in pointing some of the work in certain directions.

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SECTION 1. INTRODUCTION

For a structure to successfully satisfy the needs for which it is designed, the structural analyst/designer is often required to predict its response under a variety of dynamic loading conditions. Such a capability to predict the response of a large complex structure hinges about our ability to create suitable models, and to be able to improve and validate them through experimentation and testing. This process of the creation of an analytical model for a structural system along with procedures to experimentally test, upgrade and validate in some sense the model is what shall be referred to in this report by the term **structural identification**. Thus the identification process may be seen as a necessary prelude to the active control of structures -- control required, for instance, to suppress vibrations, achieve fine pointing in astronomical space telescopes, maintain precise shape, limit structural stresses caused by certain types of dynamic loads, or satisfy certain tolerance requirements for the adequate operation of certain subcomponents and precision instrumentation.

As such the reason why the concept of structural identification is necessary and is gaining increased importance as we continue to build large, complex, one-of-a-kind structures is our fundamental inability to analytically model, with desired precision and confidence, the structural dynamics of large, especially flexible, structures. The phases of model building, experimentation design, algorithm development and validation procedures are actually interactive, each dependent on the other, and all in turn dependent on the final needs which the desired structure is meant to satisfy. This interactive nature of the structural identification process therefore requires experimental design to be fully cognizant of: (a) the analytical model to be used (e.g., linear or nonlinear models), (b) the nature of the estimation algorithm to be used, (c) the validation criteria,

and (d) the overall goals and needs that the structure is designed to meet.

This report deals with the development of a methodology for the optimal placement of sensors in structural systems for system parameter identification(SPID) and some associated aspects of this general problem area. It may therefore be thought of as being primarily related to the experimental design aspects of the SPID process. Yet, as we shall see, the interaction with the other three phases of the process, namely, algorithm development, validation and model building is intense and cannot be neglected.

1.1. Motivation.

Millions of dollars are being spent in various industries related to the fields of aerospace, mechanical, civil and marine engineering for the experimental testing and validation of structural systems in order to ascertain that they behave under dynamic loads in a manner that they are supposedly designed to. Such experimental procedures, in essence, are rather simple. They constitute the application of known dynamic forces to the structural system and the measurement of its response at one or more locations. Yet the procedure is extremely expensive especially when carried out on full-scale systems, such as, spacecrafts and building structures. To gain the type of information that would provide the best possible model for a structural system (and thereby provide the capability to predict its response to a variety of dynamic forces that the system may be subjected to during its useful life) one therefore needs to design these experiments appropriately so that a maximum amount of information can be obtained about the relatively unknown aspects of the model through the recorded force and response time histories of motion. *Though this aspect of experimental design has been long recognized by experimentalists and structural analysts, little, if*

any, work has so far been done in this area. In fact a literature search shows but a few journal references[1] on developing a methodology for optimally locating sensors in a dynamic structural system so that records obtained from those locations would be maximally beneficial in identifying the structural system.

For large flexible structures, which are extended in space, it may not be altogether easy to ascertain where to locate sensors to obtain maximal information from the records obtained. The decision on where to locate sensors during dynamic testing of a structure is generally done today by experimentalists and technicians, and is based on experience, intuition and convenience in acquiring data. It is unfortunately not based on any cogent theoretical analysis of the situation. It is such an analysis that this report attempts to include. As we shall show in this report, the optimal sensor locations may be rather nonintuitive to see -- a consequence of our lack of experience with inverse problems. Without any well defined methodology for locating sensors, so that data from those locations will be maximally useful in system identification, often, experiments which yield little or no useful information about those aspects of the structural model that the analyst is greatest in doubt of, are carried out. This in turn necessitates more experiments, all adding up to increased costs in terms of financial and human resources, each new experiment being designed on the basis of experience and intuition rather than on some discernible rational methodology. Such an ad hoc experimental design process is not only extremely wasteful, but does not ensure that we are doing the best we possibly can. No confidence is provided in the results from a particular test in terms of its information content vis-a-vis an unknown model parameter which the test aims at assessing. Thus not only would such testing be uneconomical in terms of time and money but it would lead to erroneous results, and worse still, a lack of knowledge regarding how badly off (or erroneous) these results are.

This study has been motivated by the following types of questions, none of which so far have been adequately answered (several have not even been addressed) in the literature, on a rational basis.

- Given M sensors, where should they be placed in a spatially extended dynamic structural system so that the unknown system parameters can be "best" identified ?

- Given that M sensors are in place in a structure, where should the next N be placed so that the M+N sensors will yield maximal information about the unknown system parameters ?

- What is the minimum number of sensors required in a given structural configuration to be able to **uniquely** identify a certain set of unknown parameters ?

- How does one compress data in the recorded time series during an experiment without losing information about the parameters which need to be identified in a dynamic system ?

- Are there any trade offs between identification and control in so far as the placement of sensors is concerned ? Are there sensor locations that would be maximally beneficial for control while being minimally beneficial for identification ?

1.2. Brief Outline of the Report.

While it has been generally recognized that the quality of parametric identification of structural systems depends on the location at which sensors are placed and data gathered, very little by way of a rational methodology has so far been developed for optimally locating sensors. This report presents some new results in this direction and points out the need for more work -- work that is sorely needed if the costly phase of experimentation is to yield quality data which can be

used to best identify structural systems. While we do not pretend that we have entirely solved the five questions posed in the previous section, several new results pertaining to them have been obtained. In what follows we provide a bird's eye view of the topics considered in each section of this report, sacrificing at times exactitude and detail (which may be abundantly found in the material contained in the individual sections themselves), to provide a general sense of flow and direction to the entire report.

Section II begins with a study of uniqueness in the identification of a specific structural system which is commonly met with in aircraft and space-craft structures and more generally in the fields of aerospace, mechanical, civil and marine engineering -- the bending cantilever beam. Thus we look here at situations akin to those posed by the third question in Section 1.1. The beam is modelled in the standard manner, using finite elements and its mass distribution is assumed to be known. It is shown that for such an undamped bending beam model, the stiffness distribution throughout the beam can be uniquely determined through the placement of only two sensors when the beam is subjected to a base motion. In fact if the displacement and rotational time histories of any two adjacent nodes is available, unique identification of the entire stiffness distribution of the beam to one side of the fixed support is possible. While this may be thought of as an extension of work done in the past on shear beams[2], the results obtained here will find wider applicability because of the ubiquitous nature of bending beams, especially in space structures. While these results are predicated on the basis that there is no measurement noise, they indicate an important fact, namely that sufficient information to uniquely identify the stiffness distribution is available through the use of just two sensors, appropriately placed. Also, less than two would lead to non-unique identification, as would the use of two or more sensors if placed improperly along the cantilever. As most identification algorithms are iterative, convergence of such schemes to a

particular parameter value may not guarantee that we have found the correct system parameter unless we start off with our initial guess sufficiently close to the unknown value. The objective function surface has several local minima in which such an iterative scheme could get trapped. The significance of these uniqueness results lies in the fact that they show that one would converge to the correct parameter values starting from <u>any</u> initial guess should we appropriately locate our sensors; furthermore, only two sensors are required. The novelty in this result lies in that even the forward solution of this problem cannot be obtained in closed form.

Section III deals with the development of a methodology for optimal placement of sensors when one is interested in obtaining the "best" estimate of the unknown parameters starting from a suitable close-by initial guess and using noisy measurement data. Thus this section deals with the local optimization problem where one is attempting to reach the minimum of an objective function having started off from a close enough initial guess. Loosely speaking, we attempt to answer the following question: a) Given that we are attempting to fit noisy, measured data to our model and thereby determine the requisite unknown parameters of the model, and b) that we are hunting for the best parameters that fit the data within a small neighborhood of our a priori knowledge of the parameters, c) where should we place sensors so that a minimum covariance estimate of the parameters can be obtained ? Clearly, these locations would be partially dictated by the estimator and the estimation algorithm used and therefore the sensor locations would be estimator dependent. To get around this major difficulty, we decouple the estimation problem from the optimization problem by invoking the concept of an efficient estimator, thereby making the methodology independent of the estimator used. Thus the methodology which is developed answers questions akin to the fist two posed in Section 1.1. In fact, as we shall see, it provides a very comprehensive picture on optimal sensor locations for parametric identification in a noisy measurement

environment.

Section IV deals with an important aspect of experimental design. It attempts to throw light on the ways in which one can minimize measurement data handling and processing in SPID without relinquishing information contained in noisy data about the unknown parameters which one needs to identify from the measurements. The study is of a preliminary nature and deals with a simple single degree-of-freedom system. It has two important outcomes. Firstly, it debunks the myth that "the larger the set of data you collect (e.g. frequency data) on the response of a dynamic system the better off you are in identifying its unknown parameters." In fact, we analytically show the existence of frequency data that would have no bearing on reducing the uncertainty of the parameter estimates and therefore while being burdensome in handling and processing would cause no improvement in the results of SPID. Such studies are rudimentary at this point, but will indeed gain importance in areas such as in-flight SPID where it may be important to utilize data efficiently or in circumstances where data needs to be beamed down for terrestrial processing.

Section V deals with the tradeoffs between identification and control. This is an important aspect of SPID as it is subsumed throughout this report that one of the major reasons for improving our SPID capability is so that we can control large, flexible structures to perform in ways that are desirable and perhaps meet certain specific needs. While several investigators have alluded to the duality between identification and control on a qualitative basis, few, if any, have provided any quantitative analysis of the situation. In this section we do this through the intermediary concept of an optimal input. We investigate the following question: Given that we have only a limited amount of energy to create a forcing function to input into a dynamic system, and we create the optimal forcing function(s) to identify/control the system, what are the tradeoffs between our ability to

identify the system as opposed to our ability to control the system. The method relies on the construction of an objective function which changes continuously from one that emphasizes control to one that emphasizes identification. The formulation for a general linear dynamic structural system results in the setting up of a nonlinear two point boundary value problem. While the method developed here is applicable to structures modelled as general linear dynamic systems we have provided in this report numerical results for only a single degree-of-freedom system.

In all the abovementioned four areas, a lot remains to be done. In fact it may be fair to say that this effort barely scratches the surface of the deep seated problems that need to be faced, identified and solved if SPID of large, flexible structures can be carried out in a fruitful and reliable manner. Yet, each section of this report contains several new and useful results.

In years to come the role of SPID will indeed increase; for the determination of models of complex, flexible structures will become an important aspect of diverse application areas like control of structures, damage assessment of structures, early fault detection, and the monitoring of changes in structural systems either as a consequence of environmental conditions or as a consequence of the addition or removal of subassemblages. An excellent paradigm to keep in mind would be spacecraft structures, which need to be controlled, whose configurations may be greatly changed through docking operations with other space structures, whose materials may degrade in time due to say photon radiation, and for which early fault detection may be highly beneficial.

While several of the chapters deal with general linear dynamic systems, and develop generally applicable methodologies to answer questions such as those posed in Section 1.1 for such systems, constraints of time and resources have caused us to limit, for the purposes of this study,

our numerical studies to only the simplest numerical examples. Yet these simple problems for which numerical results have been presented in this report are perhaps quintessential to understanding the physics of the phenomena, without which insight into the behavior of complex systems would be hard to acquire. More work is clearly called for. Hopefully, the results reported herein will form a basis for perhaps a more protracted and deeper follow-on study which could include many of the major issues dealt with here. Accordingly, Chapter VI addresses the issues of what this study has taught us, in which directions further work and analyses are required, what new areas have been uncovered and what an integrated research plan addressing some of these issues should endeavor to pursue.

SECTION 2. UNIQUENESS IN THE IDENTIFICATION OF A CANTILEVER BENDING BEAM

2.1. Introduction

The identification of large structural systems is an area of research that has, over the last few years, continued to attract a significant amount of interest all over the world. Most such methods of parametric identification rely on iterative algorithms which endeavor to successively reduce the mismatch between the measured responses at one or more points in the physical system, and those predicted by a system model which purports to adequately represent the physical structure. For a given time history of measured inputs, applied at suitable locations in the structure, the system model parameters are adjusted (usually automatically) so that the time history of the mismatch is reduced as far as is computationally possible. While such a time history matching scheme will often converge to a set of model parameters, there is no guarantee, in general, that the parameters so obtained, correctly represent the actual physical system under consideration. The "zeroing-in" of the iterative scheme to an erroneous, albeit physically reasonable, set of parameters has great importance from a practical standpoint; for, this could lead to substantial errors in the calculation of quantities of engineering significance like bending moments and shear forces within the structure [3].

In a previous study [2] it was shown that a structure modelled as a discrete cantilever shear beam subjected to excitation at its fixed end (base) exhibits a degree of nonuniqueness in its identified stiffness distribution which is dependent on the location at which the time history of the response is gathered, and subsequently used for history matching. For instance, matching of the

displacement response at the mass level closest to fixed end (first mass level) was shown to cause the iterative schemes to zero-in on the correct stiffness distribution throughout the structure. On the other hand, history matching of the response gathered at the N^{th} mass level of an N mass system could lead to N! different stiffness distributions, all of which would be indistinguishable in so far as their "base input - N^{th} mass response" pairs are concerned. This realization has indeed had a significant impact on the experimental testing of spatially extended structural systems; for the placement of sensors can critically affect our confidence in the identified parameters obtained through history matching techniques.

Whereas several cantilevered structural systems used in aerospace, civil and mechanical engineering can be modelled as shear beams, several others may be more appropriately modelled by bending beams. Cantilevered bending beams are widely used in buildings, bridges, peers, offshore structures, mechanical components and subassemblages, marine vessels, nuclear reactors, aircraft and spacecraft structures and in geotechnical engineering. Hence such beams whose cross-sections and/or material properties, in general, vary along their lengths constitute structural systems that are generic in character and therefore, from a practical standpoint, they are extremely important. This section concerns itself with this category of structureal systems.

We consider a finite element model (FEM) of a cantilever undamped bending beam, and investigate the conditions under which unique identification of the stiffness of each finite element throughout the length of the beam can be achieved. The mass distribution is assumed to be known a priori throughout the structure, presumably from design drawings. In this paper we consider structural systems subjected to excitations (displacement and/or rotational inputs) at their fixed ends. It is shown that unique identification of the stiffness parameters involved in the finite element

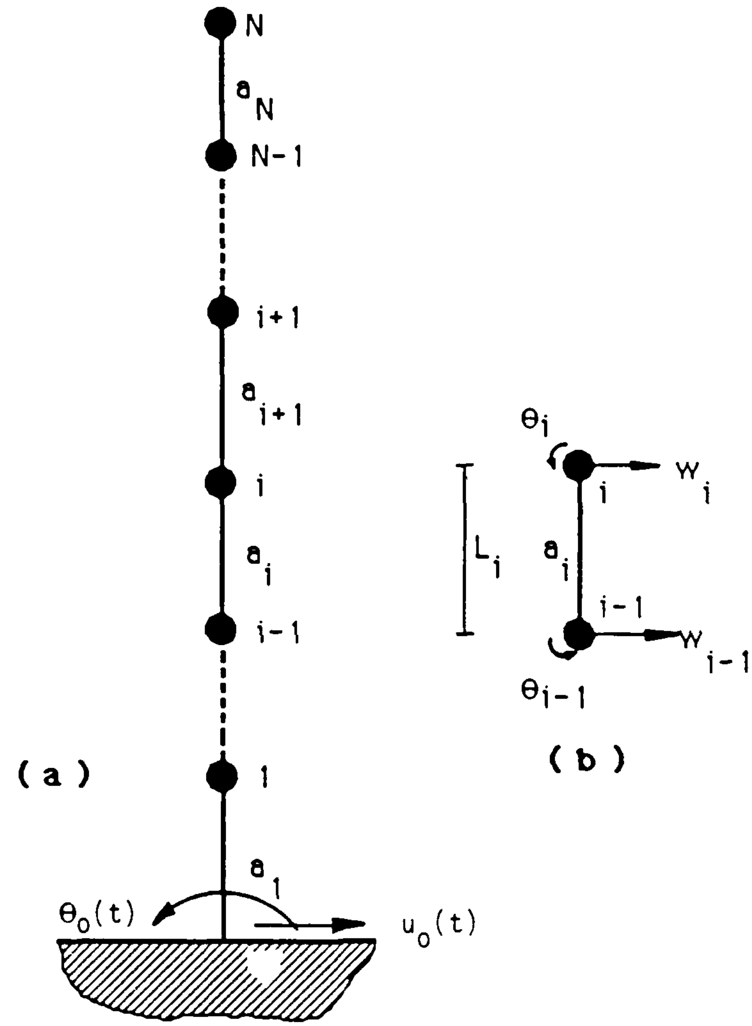


Figure 1. (a) A finite element model of the cantilever bending beam (b) a typical element of the beam.

model can be done by using data obtained at the node level closest to the fixed end. Generalizations to uniquely identifying some, though not all, of the stiffness parameters in the FEM model are also provided. The wide usage of cantilever bending beams in various fields of engineering makes these results obtained here applicable to a wide set of application areas.

2.2. System Model

Consider a structural system modelled as a linearly elastic, discrete, 2N degree of freedom bending beam as shown in Figure 1(a). The length of the i^{th} finite element is taken to be L_i and its flexural characteristic to be $(EI)_i$ (where E refers to the modulus of elasticity and I to the moment of inertia). We shall assume that both the mass and the rotary inertia at each node are known apriori as are also the lengths L_i throughout the model. Using the sign convention shown in Figure l(b) the stiffness matrix for the i^{th} element can be expressed as [4]

$$k_i = a_i \begin{bmatrix} 6 & 3L_i & -6 & 3L_i \\ 3L_i & 2L_i^2 & -3L_i & L_i^2 \\ -6 & -3L_i^3 & 6 & -3L_i \\ 3L_i & L_i^2 & -3L_i & 2L_i^2 \end{bmatrix} \tag{1}$$

where $a_i = 2(EI)_i/L_i^3$. The dynamical equations of motion can then be written as

$$\hat{M}\ddot{v} + \hat{K}v = \hat{f} \tag{2}$$

where $\hat{M} = \mathrm{diag}(m_1, m_2,...m_{2N})$ is the 2N x 2N lumped mass matrix, containing the mass and

Figure 2. Stiffness Matrix K_S

rotary inertias, $\hat{K}$ is the assembled stiffness matrix and $\hat{f}$ is the vector of external loads. The 2N-vector, v, is defined as

$$v = [W_1, \theta_1, W_2, \theta_2, ... W_N, \theta_N]^T \tag{3}$$

where W_i, θ_i are the absolute displacement and rotation at node i. The elements m_{2i}, i = 1, 2,...N correspond to the rotary moments of inertia. Since the mass matrix is positive definite, equation (1) can be renormalized to

$$\ddot{z} + Kz = f \tag{4}$$

where $z = \hat{M}^{1/2}v$, $K = \hat{M}^{-1/2}\,\hat{K}\hat{M}^{1/2}$, and $f = \hat{M}^{-1/2}\,\hat{f}$. When the excitation to the system is provided by displacements and/or rotations at the fixed end, the equations of motion of the system above the (s-1)th node level can be expressed as

$$\ddot{u} + K_s\, u = f \tag{5}$$

where $K_S = [c_{i,j}]$ is obtained by deleting the first 2s-2 rows and colums of K, and u is obtained by deleting the first 2s-2 components of v. The elements of K_s are shown in Figure 2. The (2N-2s+2)-vector, g, can be expressed as

$$g = [p_s(t), -q_s(t), 0, 0, ...0]^T \tag{6}$$

where,

$$p_s(t) = \frac{6a_s u_{s-1}(t)}{(m_{2s-1}m_{2s-3})^{1/2}} + \frac{3a_s L_s \theta_{s-1}(t)}{(m_{2s-1}m_{2s-2})^{1/2}} \quad , \tag{7a}$$

and

$$q_s(t) = \frac{3a_s L_s u_{s-1}(t)}{(m_{2s}m_{2s-3})^{1/2}} + \frac{a_s L_s^2 \theta_{s-1}(t)}{(m_{2s}m_{2s-2})^{1/2}} \tag{7b}$$

with $m_0 = m_{-1} = 1$.

The identification procedure requires the determination of a_i, $i = 1, 2,...,N$ given the knowledge of the matrix M, $\theta_0(t)$ and $u_0(t)$. We shall assume that the system start from rest so that $\dot{v}(0) = v(0) = 0$. The masses m_i, $i = 1,2,...,2N$ and the elements L_i, $i = 1, 2,..., N$ will be assumed to be known and positive. Physical interpretation of the results will be provided as we go along.

2.2.1. Some Useful Results

In this section we prove some results that will be used subsequently. The first two lemmas follow from the positive definite nature of the matrix K_s. We provide them here for the sake of completeness and also for setting out notation which we will make use of later on.

Lemma 1: Given the matrix L where

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{8}$$

in which D is invertable,

$$\det(L) = \det(D)\det(A - BD^{-1}C) \quad . \tag{9}$$

Proof: By block Gaussian elmination

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} W & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \tag{10}$$

so that $\det(L) = \det(W)\det(Z)$. Noting (10), $Z = D$ and $W = A - BD^{-1}C$ and the result follows.

Lemma 2: The matrix K_s has no zero eigenvalue. In fact

$$\det(K_s) = 3^{N-s+1} \prod_{k=s}^{N} \left(\frac{a_k^2 L_k^2}{m_{2k-1} m_{2k}} \right) > 0 \tag{11}$$

Proof: We can express K_s as

$$K_s = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{12}$$

where

$$D = \begin{bmatrix} \dfrac{6a_N}{m_{2N-1}} & \dfrac{-3a_N L_N}{m_{2N-1}^{1/2}\, m_{2N}^{1/2}} \\[2ex] \dfrac{-3a_N L_N}{m_{2N-1}^{1/2}\, m_{2N}^{1/2}} & \dfrac{2a_N L_N^2}{m_{2N}} \end{bmatrix}, \tag{13}$$

Using Lemma 1, $\det K_s = \det(D)\det(K'_{s-1}) = \dfrac{3a_N^2 L_N^2}{m_{2N} m_{2N-1}} \det(K'_{s-1})$ (14)

where K'_{s-1} is the (2N-2(s-1)-2) x (2N-2(s-1)-2) matrix obtained by deleting the last two rows and columns of K_s corresponding to a system from mass m_{2s-1} to m_{2N-2}. Using (14) successively, the result follows.

Let $P_{i,j}$ denote the matrix formed by deleting the i^{th} row and j^{th} column of $[K_s-\lambda I]$. Let $q_{i,j} = \det[P_{i,j}]$. Let $P_{0,0} = (K_s-\lambda I)$ and $R(\lambda) = \det(P_{0,0})$. The we have the following result.

Lemma 3:

$$\underset{\lambda\to\infty}{\mathrm{Lt}} \frac{\lambda q_{i,j}(\lambda)}{R(\lambda)} = \begin{cases} -1, & i = j \\ 0, & i \neq j \end{cases}, \quad \text{and} \tag{15}$$

$$\mathop{Lt}_{\lambda \to \infty} (q_{i,j}/q_{i,i}) = 0 \tag{16}$$

Proof: From the definition of $q_{i,j}(\lambda)$ and $R(\lambda)$ we have,

$$R(\lambda) = (-\lambda)^{2N-2s+2} + T(-\lambda)^{2N-2s+1} + \dots , \tag{17}$$

$$q_{i,i}(\lambda) = (-\lambda)^{2n-2s+1} + (T-c_{i,i})(-\lambda)^{2N-2s} + \dots, \tag{18}$$

$$q_{i,j}(\lambda) = (-1)^{i+j-1} c_{j,i}(\lambda)^{2N-2s} + \alpha(-\lambda)^{2N-2s-1} + \dots, i \neq j \tag{19}$$

where $T = \text{trace}(K_s)$, $[K_s] = [c_{i,j}]$, and α is a constant which can be obtained. Taking the limits the result follows.

2.3. Identification of stiffness distribution of finite element model

Using relation (5) and taking Laplace Transforms, and replacing the transform variables by $i\sqrt{\lambda}$ we get

$$(K_s - \lambda I)U(\lambda) = G(\lambda) \tag{20}$$

where $U(\lambda)$ and $G(\lambda)$ represent transformed quantities and I is the identity matrix. Solving now for $U_s(\lambda)$ and $\theta_s(\lambda)$ we get

$$U_s(\lambda) = \frac{p_s(\lambda)q_{1,1}(\lambda) + q_s(\lambda)q_{2,1}(\lambda)}{R(\lambda)}, \tag{21}$$

and

$$\theta_s(\lambda) = \frac{-[p_s(\lambda)q_{1,2}(\lambda) + q_s(\lambda)q_{2,2}(\lambda)]}{R(\lambda)} \quad . \tag{22}$$

Denoting

$$A_{s-1}(\lambda) \triangleq \frac{1}{a_s} \frac{p_s(\lambda)}{U_{s-1}(\lambda)} \quad , \quad s = 1,2,\ldots,N \tag{23}$$

$$B_{s-1}(\lambda) \triangleq \frac{1}{a_s} \frac{q_s(\lambda)}{U_{s-1}(\lambda)} \quad , \quad s = 1,2,\ldots,N \tag{24}$$

and using (7) we get

$$A_{s-1}(\lambda) = \frac{6}{(m_{2s-1}m_{2s-3})^{1/2}} + \frac{3L_s}{(m_{2s-1}m_{2s-2})^{1/2}} \frac{\theta_{s-1}(\lambda)}{U_{s-1}(\lambda)} \ , \tag{25}$$

and,

$$B_{s-1}(\lambda) = \frac{3L_s}{(m_{2s}m_{2s-3})^{1/2}} + \frac{L_s^2}{(m_{2s}m_{2s-2})^{1/2}} \frac{\theta_{s-1}(\lambda)}{U_{s-1}(\lambda)} \ , \ s = 1,\ldots,N \ . \tag{26}$$

When the $\operatorname*{Lt}_{\lambda \to \infty} A_{s-1}(\lambda)$ exists, it shall be denoted by $\overline{A}_{s-1}$. Similarly, $\overline{B}_{s-1} = \operatorname*{Lt}_{\lambda \to \infty} B_{s-1}(\lambda)$,

when the limit exists. Clearly the above limits exist if and only if

$$\left| \underset{\lambda \to \infty}{\mathrm{Lt}} \frac{\theta_{s-1}(\lambda)}{U_{s-1}(\lambda)} \right| = \gamma_{s-1} < \infty \ . \tag{27}$$

Lemma 4: If $\overline{A}_{s-1}$ and $\overline{B}_{s-1}$ exist, then they cannot both be zero.

Proof: Using the definitions of $\overline{A}_{s-1}$ and $\overline{B}_{s-1}$ the result follows.

Lemma 5: Assume that $\overline{A}_{s-1}$ and $\overline{B}_{s-1}$ exist, and $m_{2s}, m_{2s-1}, L_{s+1} > 0$. Then at least one of the following two conditions must be satisfied:

$$\text{(a)} \quad \overline{A}_{s-1} \neq 0 \ \text{ and } \ \frac{\overline{B}_{s-1}}{\overline{A}_{s-1}} \neq \frac{2}{L_{s+1}} \left(\frac{m_{2s}}{m_{2s-1}} \right)^{1/2} \tag{28}$$

$$\text{(b)} \quad \overline{B}_{s-1} \neq 0 \ \text{ and } \ \frac{\overline{A}_{s-1}}{\overline{B}_{s-1}} \neq \frac{2}{3} L_{s+1} \left(\frac{m_{2s-1}}{m_{2s}} \right)^{1/2} \ . \tag{29}$$

Proof: We show that when condition (a) is not satisfied condition (b) is always satisfied.

(i) Let $\overline{A}_{s-1} = 0$; then condition (a) is not satisfied. But by Lemma 4, $\overline{B}_{s-1} \neq 0$ also $\overline{A}_{s-1} / \overline{B}_{s-1} = 0 \neq 2/3\, L_{s+1}(m_{2s-1}/m_{2s})^{1/2}$ and condition (b) is satisfied.

(ii) Let $\overline{A}_{s-1} \neq 0$ but $\overline{B}_{s-1} / \overline{A}_{s-1} = 2/L_{s+1}\, (m_{2s}/m_{2s-1})^{1/2}$. Again condition (a) is not

satisfied. But this implies $\overline{B}_{s-1} \neq 0$ and $\overline{B}_{s-1} / \overline{A}_{s-1} \neq 3/2 L_{s+1}(m_{2s}/m_{2s-1})^{1/2}$, so that condition (b) is satisfied. Similarly, we can show that if condition (b) is not satisfied, condition (a) will always be satisfied.

Lemma 6: Given m_j, $j = 2s\text{-}3,\ldots 2s$, and L_j, $j = s, s+1$ and $U_j(\lambda)$, $\theta_j(\lambda)$, $j = s\text{-}1, s$; $\forall\lambda$, the parameters a_s and a_{s+1} can be uniquely identified under the following conditions

(1) $\overline{A}_{s-1}$ and $\overline{B}_{s-1}$ exist (30)

(2) Either

$$\text{(a)} \quad \overline{A}_{s-1} \neq 0 \text{ and } \frac{\overline{B}_{s-1}}{\overline{A}_{s-1}} \neq \frac{2}{L_{s+1}}\left(\frac{m_{2s}}{m_{2s-1}}\right)^{1/2} \tag{31}$$

$$\text{or} \quad \text{(b)} \quad \overline{B}_{s-1} \neq 0 \text{ and } \frac{\overline{A}_{s-1}}{\overline{B}_{s-1}} \neq \frac{2}{3} L_{s+1}\left(\frac{m_{2s-1}}{m_{2s}}\right)^{1/2} \tag{32}$$

Proof: We first observe (Lemma 5) that when condition (1) above is satisfied, at least one of the two conditions 2(a), 2(b) must always be satisfied. Using (21) to (24), we have

$$\frac{U_s(\lambda)}{U_{s-1}(\lambda)} = \frac{a_s}{R(\lambda)}\left(A_{s-1}(\lambda)q_{1,1} + B_{s-1}q_{2,1}\right) \tag{33}$$

$$\frac{\theta_s(\lambda)}{U_{s-1}(\lambda)} = - \frac{a_s}{R(\lambda)} \;(A_{s-1}q_{1,2} + B_{s-1}q_{2,2}) \;. \tag{34}$$

We divide the proof into two steps.

Step 1. Determination of a_s

If $\overline{A}_{s-1} \neq 0$, then by (33)

$$a_s = \left[\frac{1}{\dfrac{\lambda q_{1,1}}{R(\lambda)}}\right]\left[\frac{\lambda U_s(\lambda)}{A_{s-1}(\lambda)U_{s-1}(\lambda)}\right]\left[\frac{1}{1 + \dfrac{B_{s-1}}{A_{s-1}}\dfrac{q_{2,1}}{q_{1,1}}}\right]. \tag{35}$$

Noting Lemma 3 we have

$$a_s = - \frac{1}{\overline{A}_{s-1}} \operatorname*{Lt}_{\lambda \to \infty}\left[\frac{\lambda U_s(\lambda)}{U_{s-1}(\lambda)}\right]. \tag{36}$$

Similarly, if $\overline{B}_{s-1} \neq 0$, using (34) and the results of Lemma 3,

$$a_s = \frac{1}{\overline{B}_{s-1}} \operatorname*{Lt}_{\lambda \to \infty}\left[\frac{\lambda \theta_s(\lambda)}{U_{s-1}(\lambda)}\right]. \tag{37}$$

Note by Lemma 4 that both $\overline{A}_{s-1}$ and $\overline{B}_{s-1}$ cannot simultaneously be zero.

Step 2. Determination of a $_{s+1}$

(i) Condition 2(a) satisfied:

Expanding $[K_S-\lambda I]$ in terms of the cofactors of the first column we have,

$$\frac{R(\lambda)}{q_{1,1}(\lambda)} = \left(\frac{6a_s + 6a_{s+1}}{m_{2s-1}} - \lambda\right) - \frac{(3a_{s+1}L_{s+1}-3a_sL_s)}{m_{2s}^{1/2}m_{2s-1}^{1/2}}\frac{q_{2,1}}{q_{1,1}} \tag{38}$$

$$-\frac{6a_{s+1}}{(m_{2s+1}m_{2s-1})^{1/2}}\frac{q_{3,1}}{q_{1,1}} - \frac{3a_{s+1}L_{s+1}}{(m_{2s+2}m_{2s-1})^{1/2}}\frac{q_{4,1}}{q_{1,1}}$$

$$\stackrel{\Delta}{=} \left(\frac{6a_s + 6a_{s+1}}{m_{2s-1}} - \lambda\right) + D(\lambda) \ .$$

Also by (21) and (23)

$$\frac{R(\lambda)}{q_{1,1}(\lambda)} = a_s\frac{U_{s-1}(\lambda)}{U_s(\lambda)}\left[A_{s-1} + B_{s-1}\frac{q_{2,1}}{q_{1,1}}\right] \ . \tag{39}$$

By Lemma 3, $\underset{\lambda\to\infty}{\mathrm{Lt}}\ D(\lambda) = 0$. Also, for $\overline{A}_{s-1} \neq 0$, using (33) and Lemma 3,

$$\underset{\lambda\to\infty}{\mathrm{Lt}}\left[a_sB_{s-1}\frac{U_{s-1}}{U_s}\frac{q_{2,1}}{q_{1,1}}\right] = \underset{\lambda\to\infty}{\mathrm{Lt}}\left[a_s\frac{B_{s-1}}{A_{s-1}}\frac{A_{s-1}U_{s-1}(\lambda)}{\lambda U_s(\lambda)}\frac{\lambda q_{2,1}}{q_{1,1}}\right] \tag{40}$$

$$= 3\,\frac{\overline{B}_{s-1}}{\overline{\overline{A}}_{s-1}}\left\{\frac{a_{s+1}L_{s+1} - a_s L_s}{(m_{2s}m_{2s-1})^{1/2}}\right\}.$$

Equating the right hand sides of equations (38) and (39) and taking limits as $\lambda \to \infty$, we get

$$a_{s+1}\left[\frac{2}{m_{2s-1}^{1/2}} - \frac{L_{s+1}\overline{B}_{s-1}}{m_{2s}^{1/2}\,\overline{A}_{s-1}}\right]$$

$$= \frac{(m_{2s-1})^{1/2}}{3}\,\underset{\lambda\to\infty}{\mathrm{Lt}}\left[\lambda + a_s A_{s-1}\,\frac{U_{s-1}(\lambda)}{U_s(\lambda)}\right]$$

$$- a_i\left[\frac{2}{m_{2s-1}^{1/2}} + \frac{L_s\overline{B}_{s-1}}{m_{2s}^{1/2}\,\overline{A}_{s-1}}\right]. \tag{41}$$

The result therefore follows.

(ii) Condition 2(b) satisfied:

The determination of a_{s+1} is similar to the above. We therefore only sketch the proof briefly. Expand $R(\lambda)$ in terms of the cofactors of the second column of the matrix $[K_s - \lambda I]$ and divide by $q_{2,2}$. Also using (22) and (24) we get

$$\frac{R(\lambda)}{q_{2,2}(\lambda)} = -a_s A_{s-1} \frac{U_{s-1}(\lambda)}{\theta_s(\lambda)} \frac{q_{1,2}(\lambda)}{q_{2,2}(\lambda)} - a_s B_{s-1} \frac{U_{s-1}(\lambda)}{\theta_s(\lambda)} . \tag{42}$$

Equating the two expressions for $R(\lambda)/q_{2,2}(\lambda)$ and taking limits as $\lambda \to \infty$, we get

$$a_{s+1}\left[\frac{2L_{s+1}}{m_{2s}^{1/2}} - \frac{3}{m_{2s-1}^{1/2}} \frac{\overline{A}_{s-1}}{\overline{B}_{s-1}}\right] = \frac{m_{2s}^{1/2}}{L_{s+1}} \operatorname*{Lt}_{\lambda\to\infty} [\lambda - a_s B_{s-1} \frac{U_{s-1}}{\theta_s}]$$

$$- \frac{a_s L_s}{L_{s+1}}\left[\frac{\overline{A}_{s-1}}{\overline{B}_{s-1}} \frac{3}{m_{2s-1}^{1/2}} + \frac{2L_s}{m_{2s}^{1/2}}\right] \tag{43}$$

and hence the result.

Defining

$$A'_{s-1}(\lambda) \triangleq \frac{1}{a_s} \frac{p_s(\lambda)}{\theta_{s-1}(\lambda)} = \frac{6}{m_{2s-1}^{1/2} m_{2s-3}^{1/2}} \frac{U_{s-1}(\lambda)}{\theta_{s-1}(\lambda)} + \frac{3L_s}{m_{2s-1}^{1/2} m_{2s-2}^{1/2}} \tag{44}$$

$$B'_{s-1}(\lambda) \triangleq \frac{1}{a_s} \frac{q_s(\lambda)}{\theta_{s-1}(\lambda)} = \frac{3L_s}{m_{2s}^{1/2} m_{2s-3}^{1/2}} \frac{U_{s-1}(\lambda)}{\theta_{s-1}(\lambda)} + \frac{L_s^2}{m_{2s}^{1/2} m_{2s-2}^{1/2}} \tag{45}$$

for $s = 1, 2, \ldots N$ with $m_0 = m_{-1} = 1$, we see that

$$\bar{A}'_{s-1} \triangleq \operatorname*{Lt}_{\lambda\to\infty} A'_{s-1}(\lambda) \text{ and } \bar{B}'_{s-1} \triangleq \operatorname*{Lt}_{\lambda\to\infty} B'_{s-1}(\lambda) \tag{46}$$

exist if and only if

$$\left| \operatorname*{Lt}_{\lambda\to\infty} \frac{U_{s-1}(\lambda)}{\theta_{s-1}(\lambda)} \right| = \gamma'_{s-1} < \infty \tag{47}$$

Using this notation, relations (21) and (22) can be expressed as

$$U_s(\lambda) = a_s \frac{\theta_{s-1}(\lambda)}{R(\lambda)} [A'_{s-1} q_{1,1} + B'_{s-1} q_{2,1}] , \tag{48}$$

and

$$\theta_s(\lambda) = -a_s \frac{\theta_{s-1}(\lambda)}{R(\lambda)} [A'_{s-1} q_{1,2} + B'_{s-1} q_{2,2}] . \tag{49}$$

We then have the following results.

Lemma 7: The quantities A_{s-1}, B_{s-1}, $A_s'{}_{-1}$ and $B_s'{}_{-1}$ defined in equations (25), (26), (44) and (45) are such that at least one of the following two conditions is always satisfied.

1. $\bar{A}_{s-1}$ and $\bar{B}_{s-1}$ exist
2. $\bar{A}s'{}_{-1}$ and $\bar{B}s'{}_{-1}$ exist .

Proof: Condition 1 occurs iff $\left| \underset{\lambda \to \infty}{\mathrm{Lt}} \dfrac{\theta_{s-1}(\lambda)}{U_{s-1}(\lambda)} \right| \leq \gamma_{s-1} < \infty$. If this condition is not

satisfied, then it follows that $\left| \underset{\lambda \to \infty}{\mathrm{Lt}} \dfrac{U_{s-1}(\lambda)}{\theta_{s-1}(\lambda)} \right| \leq \gamma'_{s-1} < \infty$, and therefore condition 2

is satisfied.

Lemma 8: Assume that $\overline{A}'_{s-1}$ and $\overline{B}'_{s-1}$ exist and $m_{2s}, m_{2s-1}, L_{s+1} > 0$.

Then at least one of the following two conditions must always be satisfied

$$\text{(a)} \quad \overline{A}'_{s-1} \neq 0 \text{ and } \frac{\overline{B}'_{s-1}}{\overline{A}'_{s-1}} \neq \frac{2}{L_{s+1}} \left(\frac{m_{2s}}{m_{2s-1}}\right)^{1/2} \tag{50}$$

$$\text{(b)} \quad \overline{B}'_{s-1} \neq 0 \text{ and } \frac{\overline{A}'_{s-1}}{\overline{B}'_{s-1}} \neq \frac{2}{3} L_{s+1} \left(\frac{m_{2s-1}}{m_{2s}}\right)^{1/2} . \tag{51}$$

Proof: The proof is similar to that of Lemma 5.

Lemma 9: Given m_j, $j = 2s-3,...2s$; L_j, $j = s, s+1$, and $U_j(\lambda)$, $\theta_j(\lambda)$, $j = s-1, s$; for all λ, then parameters a_s and a_{s+1} can be uniquely determined under the following conditions

(1) $\overline{A}'_{s-1}$ and $\overline{B}'_{s-1}$ exists, and (52)

(2) Either

$$\text{(a)} \quad \overline{A}'_{s-1} \neq 0 \text{ and } \frac{\overline{B}'_{s-1}}{\overline{A}'_{s-1}} \neq \frac{2}{L_{s+1}} \left(\frac{m_{2s}}{m_{2s-1}}\right)^{1/2} \tag{53}$$

$$\text{or} \quad \text{(b)} \quad \overline{B}'_{s-1} \neq 0 \text{ and } \frac{\overline{A}'_{s-1}}{\overline{B}'_{s-1}} \neq \frac{2}{3} L_{s+1} \left(\frac{m_{2s-1}}{m_{2s}}\right)^{1/2} . \tag{54}$$

Proof: The proof is along the same lines as that of Lemma 4. We therefore only provide the results.

(i) If $\overline{A}'_{s-1} \neq 0$

$$a_s = -\frac{1}{\overline{A}'_{s-1}} \operatorname*{Lt}_{\lambda\to\infty} \left[\frac{\lambda U_s(\lambda)}{\theta_{s-1}(\lambda)}\right], \text{ and} \tag{55}$$

$$a_{s+1} = \left[\frac{2}{m_{2s-1}^{1/2}} - \frac{\overline{B}'_{s-1}}{\overline{A}'_{s-1}} \frac{L_{s+1}}{m_{2s}^{1/2}}\right]^{-1} \left\{ \frac{m_{2s-1}^{1/2}}{3} \operatorname*{Lt}_{\lambda\to\infty} [\lambda + a_s \frac{\theta_{s-1}(\lambda)}{U_s(\lambda)} A'_{s-1}(\lambda)] - a_s [\frac{2}{m_{2s-1}^{1/2}} + \frac{\overline{B}'_{s-1}}{\overline{A}'_{s-1}} \frac{L_s}{m_{2s}^{1/2}}] \right\} \tag{56}$$

(ii) If $\overline{B}_s{}'_{-1} \neq 0$

$$a_s = -\frac{1}{\overline{B}'_{s-1}} \underset{\lambda\to\infty}{\mathrm{Lt}} \left[\frac{\lambda\theta_s(\lambda)}{\theta_{s-1}(\lambda)}\right], \text{ and} \tag{57}$$

$$a_{s+1} = \left[\frac{2}{m_{2s}^{1/2}} L_{s+1} - \frac{3}{m_{2s-1}^{1/2}} \frac{\overline{A}'_{s-1}}{\overline{B}'_{s-1}}\right]^{-1} \left\{\frac{m_{2s}^{1/2}}{L_{s+1}} \cdot \underset{\lambda\to\infty}{\mathrm{Lt}} \left[\lambda - a_s B'_{s-1} \frac{\theta_{s-1}(\lambda)}{\theta_s(\lambda)}\right]\right.$$

$$\left. - \frac{a_s L_s}{L_{s+1}} \left[\frac{\overline{A}'_{s-1}}{\overline{B}'_{s-1}} \frac{3}{m_{2s-1}^{1/2}} + \frac{2L_s}{m_{2s}^{1/2}}\right]\right\}. \tag{58}$$

We note that at any node, s-1, of the finite element model, (by Lemma 7), at least one of the conditions (30) or (52) must always be satisfied; thus at least one of the two Lemmas 6 and 9 is always applicable. We have therefore shown that if the displacement and rotational time histories of motion are known at two consecutive nodes of a FEM of a bending beam along with the mass and rotary inertia properties corresponding to those two nodes, then a unique identification of the material property of the element in between the nodes and also that of the element above the upper node can be obtained. We next show that the data obtained so far, uniquely determines the displacement and rotational time histories at the next node point.

Lemma 10: Given:

1. a_s, a_{s+1}
2. U_i, θ_i, i = s-1,s
3. L_i, i = s, s+1, and
4. m_i, i = 2s-1, 2s+2 ,

$U_{s+1}(\lambda)$ and $\theta_{s+1}(\lambda)$ can be uniquely determined.

Proof: Considering the first two rows of equation (20) and noting the structure of K_s we have

$$U_{s+1}(\lambda) = \left[X_s \frac{a_{s+1}L_{s+1}^2}{m_{2s}^{1/2} m_{2s+2}^{1/2}} + Y_s \frac{3a_{s+1}L_{s+1}}{m_{2s-1}^{1/2}\, m_{2s+2}^{1/2}} \right] / Z_s \ , \tag{59}$$

and

$$\theta_{s+1}(\lambda) \ = - \left[Y_s \frac{6a_{s+1}}{m_{2s-1}^{1/2} m_{2s-1}^{1/2}} + X_s \frac{3a_{s+1}L_{s+1}}{m_{2s}^{1/2}\, m_{2s+1}^{1/2}} \right] / Z_s \tag{60}$$

where,

$$X_s = p_s(\lambda) - (c_{1,1} - \lambda)U_s(\lambda) - c_{1,2}\theta_s(\lambda) \tag{61}$$

$$Y_s = - q_s(\lambda) - c_{1,2}U_s(\lambda) - (c_{2,2} - \lambda) \ \theta_s(\lambda) \tag{62}$$

and

$$Z_s = (3a_{s+1}^2 L_{s+1}^2) / \prod_{i=2s-1}^{2s+2} m_i^{1/2} > 0 . \tag{63}$$

Hence the result.

We next show some results related to the nature of the quantities $\overline{A}_{s-1}, \overline{B}_{s-1}, \overline{A}_s, \overline{B}_s$.

Lemma 11. If $\overline{A}_{s-1}$ and $\overline{B}_{s-1}$ exist then

(a) if $\overline{A}_s \neq 0$, $\overline{A}_s$ and $\overline{B}_s$ exist

(64)

(b) if $\overline{B}_{s-1} \neq 0$, $\overline{A}_s'$ and $\overline{B}_s'$ exist.

Proof: (a) Using (21) and (24),

$$\frac{\theta_s}{U_s} = - \frac{A_{s-1}q_{1,2} + B_{s-1}q_{2,2}}{A_{s-1}q_{1,1} + B_{s-1}q_{1,2}} . \tag{65}$$

Dividing the numerator and denominator by $\lambda/R(\lambda)$, and taking limits $\lambda \rightarrow \infty$ and using Lemma 3,

we get

$$\left| \underset{\lambda \to \infty}{\mathrm{Lt}} \frac{\theta_s(\lambda)}{U_s(\lambda)} \right| = \left| - \frac{\overline{B}_{s-1}}{\overline{A}_{s-1}} \right| < \gamma_s < \infty \, . \tag{66}$$

Using (25) and (26), thus $\overline{A}_s$, and $\overline{B}_s$ exist. The proof of part (b) is similar.

Lemma 12: If $\overline{A}'_{s-1}$ and $\overline{B}'_{s-1}$ exist then

(a) if $\overline{A}'_{s-1} \neq 0$, $\overline{A}_s$ and $\overline{B}_s$ exist (67)

(b) if $\overline{B}'_{s-1} \neq 0$, $\overline{A}_s'$ and $\overline{B}_s'$ exist.

Proof: The proof is similar to that of Lemma 11.

2.4. Main Result

Theorem 1. For the FEM of the cantilever bending beam considered, given

(1) m_j, $j = 2s-3, \ldots, 2N$

(2) L_j, $j = s, \ldots, N$, and (68)

(3) $U_j(t)$, $\theta_j(t)$, $j = s-1, s$,

the element stiffness distribution

$$a_j, \; j = s, s+1, \ldots, N$$

can be uniquely determined.

Proof: The proof is divided into three parts and being recursive can best be expressed in algorithmic form as follows.

Do steps 1 $\rightarrow$ 3, for i = s,N-1

Step (1): Find A_{i-1}, B_{i-1}, A'_{i-1}, B'_{i-1} using expressions (25), (26), (44) and (45).

Step (2): Use either Lemma 6 or Lemma 9 depending on $\bar{A}_{i-1}$ and $\bar{A}'_{i-1}$ to determine a_i and a_{i+1}. (By Lemmas 7, 5 and 8, one of these two lemmas must always be applicable.)

Step (3): Use Lemma 10 to find $U_{i+1}(\lambda)$, $\theta_{i+1}(\lambda)$.

We have thus shown that the stiffness distribution above a particular node of the FEM can be uniquely determined by obtaining the displacement and rotational time histories of motion at that node and the node directly above it. Thus the distribution can be uniquely determined by locating sensors at only two points in the system.

Corollary: The stiffness of every element of the finite element model of the cantilever bending beam can be determined given

(1) m_j, j = 1, 2,...,2N

(2) L_j, j = 1,..., N, and

(3) $U_0(t)$, $\theta_0(t)$, $U_1(t)$, $\theta_1(t)$.

Proof: Set s = 1 in Theorem 1 and the result follows.

We have thus shown that the entire finite element model of the structure can be uniquely identified if the base displacement and base rotational time histories of motion are known as well as the displacement and rotational time histories of motion at the first node of the structure. In particular, unique identification of the entire FEM model can be done if for a known base displacement time history of motion the rotational and displacement time histories of motion are provided at the first node.

2.5. Conclusions

In this section we have modelled an undamped cantilever bending beam by the finite element method, and have attempted to identify the stiffness properties of the elements. The stiffness matrix of each element is taken to be that usually used in day-to-day structural analysis and design. It is shown that knowledge of the displacement and rotational time histories of motion at the fixed end and at the node closest to the fixed end yield unique identification of the properties of each of the finite elements of the model. More generally, the knowledge of the rotational and displacement time histories of motion at any two consecutive nodes, say i and i+1 (see Figure 1), provides enough information to uniquely determine the stiffness properties of all the elements that lie to one side of the node i, away from the fixed end.

It should be noted that the analysis presented here deals with noise free data. The presence of noise in the measurements would in general lead to low signal-to-noise ratios for measurements at the lowest node levels so that though the identification problem has a unique solution, in actual practice, the variance of the parameter estimates may become large. Measurements made farther

away from the fixed end would be larger in amplitude so that the noise related estimation errors would be smaller. However, one faces then the nonuniqueness problem. Furthermore, the analysis herein assumes that a complete knowledge of the input-output time histories (or their transforms) is available.

While no attention has been given to the process of arriving at an adequate finite element model (i. e. the proper location and number of nodal points) the prediction of the response of the actual physical system to various input time histories using the identified parameters (obtained by history matching) critically depends on the adequacy of the model. Structural modelling being to a good extent an art acquired mainly by practice, this aspect of the problem has not been considered. However, it should be pointed out that the uniqueness results arrived at here are <u>independent of the number of nodes chosen</u>, i. e. no matter how many finite elements are used to represent the system (i. e. no matter how many parameters to be identified) <u>only two sets of measurements</u> (at the fixed end and at the node nearest the fixed end) are sufficient to tie down the estimates of the stiffness of each element uniquely. The analysis of damped structural models has been specifically excluded for the sake of simplicity and work along these lines is continuing.

Cantilevered bending beams form one of the commonest categories of structural elements and are used widely across diverse fields of engineering. They find application in the fields of civil, mechanical, nuclear, aerospace and marine engineering to name but a few and hence these results may have significance to a wide set of application areas. Large boom type structures in the aerospace industry, tall buildings in civil engineering and the identification of subassemblages in the nuclear industry are but a few specific examples.

SECTION 3. A METHODOLOGY FOR OPTIMUM SENSOR LOCATIONS FOR PARAMETRIC IDENTIFICATION OF DYNAMIC SYSTEMS

3.1. Introduction

Reliable predictions of structural responses are closely dependent upon the validity of the models chosen to represent the systems involved. When parametric models are used, a proper knowledge of the various parameter values becomes crucial in establishing the usefulness of such models. However, to actually come up with these parameter values, one often needs to collect response data from instruments located at various positions within the structure. The usefulness of such data, in turn, depends primarily upon the instrument characteristics and on the chosen positions where the instruments are located. Consequently, for given types of instruments, which are to be used, one often wants to locate them such that data collected from those locations yield the "best" estimates of the modelled structural parameters.

Although various methods have been developed to identify the parameters that characterize flexible structures (e. g., [5-10]), from records obtained in them under various loading conditions, few investigators, if any, have looked at the question of where to locate sensors in a large, spatially extended structure to acquire data for "best" parametric identification [1]. The problem of optimally locating sensors in a dynamic vibrating system mainly arises from considerations of: (1) minimizing the cost of instrumentation, data processing and data handling through the use of a smaller number of sensors, data channels, etc.; (2) obtaining better (more accurate) estimates of

model parameters from noisy measurement data; (3) improving structural control through the use of superior structural models; (4) efficiently determining structural properties and their changes with a view to acquiring improved assessments of structural integrity, and (5) improving the early fault detection capability for large, flexible structural systems.

The problem addressed in this section can succiently be stated as follows: Given m sensors where should they be located in a spatially distributed dynamic system so that records obtained from those locations yield the "best" estimates of those relatively unknown parameters which need to be identified?

In the past, the optimum sensor location problem (OSLP) was solved by positioning the given sensors in the system, using the records obtained at those locations with a specific estimator, and repeating the procedure for different sensor locations. The set of locations which yield the "best" parameter estimates would then be selected as optimal. The estimates obtained of course depend upon the type of estimator used. Thus the optimal locations are estimator dependent, and an exhaustive search needs to be performed for each specific estimator. Such a procedure besides being highly computationally intensive suffers from the major drawbacks of not yielding any physical insight into why certain locations are preferable to others.

Work on the solution of the OSLP was perhaps first done by Shah and Udwadia [1]. In brief, they used a linear relationship between small perturbations in a finite dimensional representation of the system parameters and a finite sample of observations of the system time response. The error in the parameter estimates was minimized yielding the optimal locations. In this section, we developed a more direct approach to the problem which is both computationally

superior, and throws considerable light on the rationale behind the optimal selection process. The methodology is applicable to all spatially extended dynamic systems. In this report, special attention has been paid to large, flexible structural systems such as large space structures.

We uncouple the optimization problem from the identification problem using the concept of an efficient estimator (e. g., the maximum likelihood estimator as the time history of data becomes very large)[11]. For such an estimator the covariance of the parameter estimates is a minimum. Using this technique and motivated by heuristic arguments, a rigorous formulation and solution of the OSLP is presented.

3.2. Uniqueness of Identification and Local Optimization

The iterative nature of most identification schemes require us to differentiate between two distinct, and often times confusing, criteria for the sensor location problem.

(1) Uniqueness of Identification

The idea here is to locate sensors in such a manner that no matter what initial guess (of the parameters or functionals) one starts off with in the iterative scheme, the identification will converge to the unique "actual" system. Alternatively put, one wants to locate sensors at locations which yield information that can unequivocally tell us the parameters of the actual system. Using records obtained from such locations, different initial guesses would not yield different estimates of the parameters (or functionals) from those of the actual system. Section 2 of this report dealt with

this aspect for a cantilever bending beam, a commonly used substructure in structural design.

Such nonuniqueness problems if they in fact did exist could indeed lead to not only incorrect identification of structural system parameters, **irrespective** of the actual estimator used, but could also lead to incorrect estimates of quantities of engineering significance like bending moments and shear forces within the structure. Such incorrect estimates would directly affect our ability to assess the structural integrity of the system and our ability to control it appropriately.

(2) Local Optimization

If however, one has a fairly good idea of the parameter values, the initial guesses would be fairly close to the actual values. Thus, provided the measurements are not too noisy, one would not be likely to converge to a solution other than that represented by the actual system. Having restricted the search space in this manner, so to speak, through the use of a good initial guess, one needs to locate sensors in such a manner that having started with these approximate estimates, the records obtained have the greatest information to improve the estimates. As opposed to the "global" convergence (starting from "any" initial guess) which Section 2 addressed, this section looks at the development of a methodology for optimally locating sensors in the context of "local" convergence (starting from a "close" initial guess). In Section 3.9.3 we will further illustrate some of the differences between these two concepts.

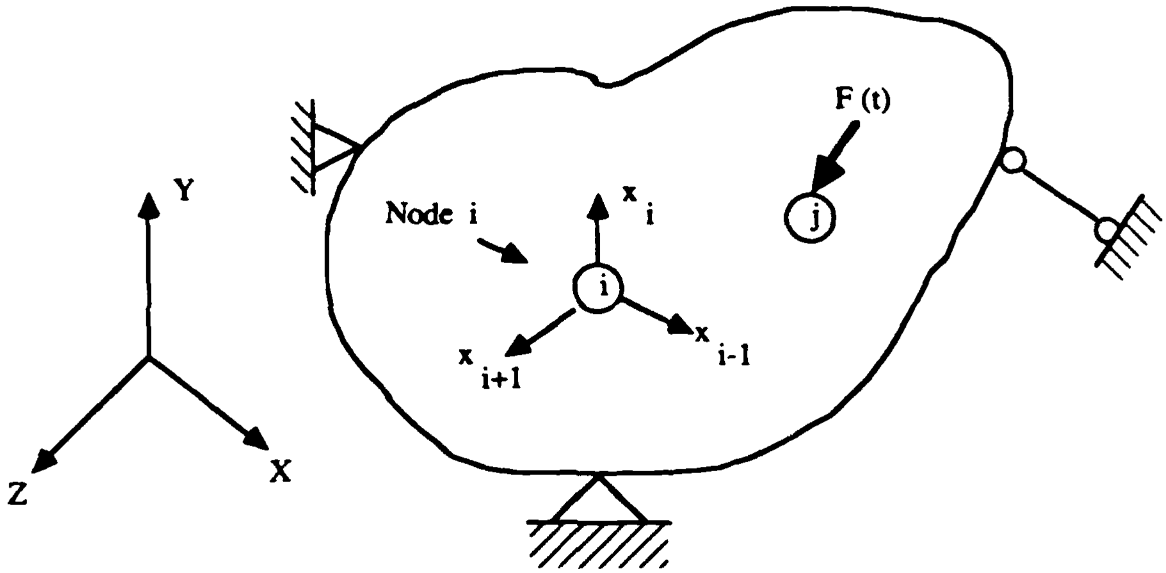


Figure 3. Nodal displacements, x_i, of a system subjected to external dynamic loading F(t).

3.3. Model Formulation

3.3.1. System Model

The study of physical systems which can be adequately represented by linear constant coefficient differential equations will be considered in this sequel. Though most large complex dynamic systems are spatially continuous in nature, often, suitable discrete models can be formulated for engineering applications. In fact finite element and finite difference methods have become common in the reduction of physically continuous systems to mathematically discrete models.

Though the development of the optimum sensor location (OSL) criterion will be shown to be unrelated to the nature (linear, nonlinear, time-variant, time-invariant) of the system S under consideration, let us for the moment consider a linear dynamic vibrating system so that we have a vehicle for developing the methodology.

The governing differential equation of motion for a linear dynamic system may be considered as :

$$M\ddot{X} + C\dot{X} + KX = F(t)\ ;\ \dot{X}(0) = \dot{X}_0\ ,\ X(0) = X_0 \tag{69}$$

where $\dot{X}_0$ and X_0 are the given initial conditions for the system. The constant coefficient matrices M, C and K are each of dimension $(N \times N)$. The M matrix may be considered as the lumped or the consistent mass matrix (as is often the case in structural analysis), C as the damping matrix and

K as the stiffness matrix. X is an N-vector whose components, x_j, may be considered to be the displacement response (at the nodes of the finite element or the finite difference mesh) of the system shown in Figure 3 to the input vector $\{F(t)\}_N$. One or more elements of the coefficient matrices, in equation (1), may constitute the unknown parameters. To "best" estimate these parameters, one would locate sensors in the system in such a way that the measurements obtained thereat are most informative about the estimated parameters. To accomplish this task, let us collect all the possible unknown parameters in a vector θ of dimension L. Hence

$$\theta = < \theta_M \mid \theta_C \mid \theta_K >^T \tag{70}$$

where the subvectors θ_M, θ_C, and θ_K have dimensions a, b, and c, respectively. The superscript T indicates vector transpose throughout this dissertation. In general, a, b, and c are each at most equal to N^2; however, in many dynamic systems, the coefficient matrices in equation (69) are symmetric. In such a case a, b, and c are each at most equal to $(N + 1)N/2$.

3.3.2. Measurement Model

The response of the dynamic system is assumed to be measured using an m $(m < N)$ available sensors. Solution of the OSLP is equivalent to the selection of the m locations out of N possible such locations so that the m time history response obtained thereat yield the maximum amount of information about the system parameters.

To formulate the measurement model, let us first assume there are exactly N sensors

available, so that each component of X is measured (i. e., $m = N$). These measured responses can be mathematically represented by the N-vector Z as follows:

$$Z_j(t) = g_j\,[X(\theta, t)\,] + N_j(t)], \quad j = 1, 2, \ldots, N \tag{71}$$

where Z_j is the j-th component of Z(t), functional g_j represents the "measurement process", and the dependence of the response X on the parameter θ is explicitly noted. We shall assume that g is a memoryless transformation of the system output which yields the measurements. The measurement noise $N_j(t)$ is taken as nonstationary Gaussian white noise with a variance of $\psi^2(t)$. Therefore,

$$E[N_j(t_1)N_j(t_2)] = \psi^2(t_1)\,\delta_K(i\text{-}j)\,\delta_D(t_1 - t_2) \tag{72}$$

where δ_K and δ_D stand for the Kroneker and the dirac-delta functions, respectively. Having measured each element of the response vector X, a total of m out of N responses need to be selected so that they contain the most information about the system parameters and are maximally sensitive to any change in the parameter values. This "selection" process can be represented by an m-dimensional vector Y such that

$$Y(t) = SZ(t) \tag{73}$$

where S is the (m x N) upper triangular selection matrix with each row containing null elements except for one which is unity. The m different components of Z selected to be measured are so

ordered in vector Y, that if the element in the i-th row and k-th column of S is unity, the (i+1)-th row can have unity in its ℓ-th column only if $\ell > k$. The matrix S then has the property so that matrix $P = S^T S$ is an (N x N) diagonal matrix with unity in its i-th row if, and only if, Z_i is selected to be measured. The elements of P are otherwise zero. Hence, one can write

$$Y(t) = Sg[X(\theta, t)] + SN(t), \tag{74}$$

or

$$Y(t) = H[X(\theta, t)] + V(t) \tag{75}$$

If g_i is linearly related to the response x_j, in general, then

$$H[X(\theta, t)] = SRX \tag{76}$$

where R(t) in equation (76) can be thought of as a dynamic gain matrix. In the case that g_i is related to the response x_i only, matrix R will reduce to a diagonal matrix, $\mathrm{Diag}(\rho_1, \rho_2, \ldots, \rho_N)$.

The probiem of locating sensors in an optimal manner then reduces to determining the selection matrix S defined above. Alternately put, one needs to determine the m locations along the diagonal of the matrix P that should be unity. These locations must be so chosen as to obtain the "best" parameter estimates.

3.4. Efficient Estimator and the Cramer-Rao Lower Bound

3.4.1. Conditional and Unconditional Estimation

In this section we shall consider two types of estimations. (1) The conditional estimation problem results when θ is an unknown constant, and/or when the a priori probability density function $P(\theta)$, is not known. For this type of problem the expected value of the parameter estimate equals the true value of the quantity being estimated and is often referred to as conditionally unbiased estimation [12,13]. (2) In the unconditional estimation problem the unknown parameter θ is characterized by a known prior probability density function. In this type of problem the expected value of the estimate equals the expected value of the quantity being estimated and is called unconditionally unbiased estimation [12,13].

Unconditional estimation is more general; however, in many situations when the probability distributions of the parameter estimates are not know, one can not utilize them. Suppose it is desired to find an estimate of the parameter vector θ; if the joint density $P_\theta(\theta)$ is not in hand or θ is not a random vector, then conditional estimation could be used.

In the analysis of flexible structures, an example of conditional estimation may involve determination of the structural parameters of a space structure from noisy observations. In such a case, the unknown parameters, say the member stiffnesses, are considered as unknown constants

which remain constant even after many observational measurements are made. An example of unconditional estimation, on the other hand, involves some knowledge of the probability density of the unknown random parameters.

3.4.2. Cramer-Rao lower bound (CRLB) and Cramer-Rao (CR) inequalities

The lower bound of the covariance of the estimation error was first proposed by Cramer and is called the Cramer-Rao Lower Bound (CRLB) [11, 14]. This CRLB is related to the minimum error between the estimates and the actual value of the parameters to be identified in the sample distributions given. The CRLB is a nontrivial lower bound on the covariance of the estimate. Unfortunately, the CRLB depends upon the bias of whatever estimator is used. However, for unbiased estimators the CRLB is indeed independent of the estimator that might be employed. For any chosen arrangement of sensor locations, expressions for the conditional CRLB (CCRLB) and the unconditional CRLB (UCRLB) for unbiased estimators are derived in [15]. They are:

$$\mathrm{CCRLB} = \left\{ \int_0^T \left(\frac{\partial H(\theta,t)}{\partial \theta} \right)^T \left(\frac{\partial H(\theta,t)}{\partial \theta} \right) / \psi^2(t) dt \right\}^{-1} \tag{77}$$

$$\mathrm{UCRLB} = \left\{ \int_{\Omega_\theta} P_\theta(\theta) \cdot (\mathrm{CCRLB})^{-1} d\theta \right\}^{-1} . \tag{78}$$

Hence

$$E[\,(\theta - \hat{\theta})\ (\theta - \hat{\theta})^T \mid \theta\,] \geq CCRLB \tag{79}$$

and

$$E[(\theta - \hat{\theta})\,(\theta - \hat{\theta})^T, \theta] \ \geq UCRLB \tag{80}$$

The right hand side of the inequalities (79) and (80) can be obtained from equations (77) and (78), their left hand sides depending on the type of estimator used. Inequalities (79) and (80) are known as the conditional and the unconditional estimation covariance inequalities.

3.4.3. Efficient estimators

If an unbiased estimator achieves the CRLB, the estimator is called efficient. An efficient estimator may also be called an optimal unbiased estimator since it achieves a minimum "Expected Square of Estimation Error" (ESEE) and no other unbiased estimator could, in fact, achieve a lower ESEE. It is of some interest to also note (see equations 79 and 80) that an unbiased, efficient estimator is also a minimum-variance estimator. It is however, unfortunate, that there are no general methods for constructing efficient estimators. But one commonly useful class of estimators which is asymptotically unbiased and efficient is the one which comprises the conditional and unconditional maximum likelihood estimators ([14],[16]). Hence for efficient estimators (minimum covariance) the inequalities [5] become equalities. Therefore, one can write

$$E[(\theta - \hat{\theta})\,(\theta - \hat{\theta})^T] = CRLB \tag{81}$$

In this sequel such as estimator shall be assumed to exist.

The expression inside the brackets in equalities (77) and (78) are known as the Fisher Information (F. I.) matrices [15]. Therefore maximization of this Fisher Information matrix (maximizing a certain norm of the matrix; such as the trace norm, etc...) would yield the minimum possible value of the covariance of the estimation error [17,18]. It is of some interest to note that when using efficient estimators, the covariance of the estimation error is known with no regard to the actual form of the estimator.

The Fisher Information Matrix, Q(T), for conditional estimation of θ (using equation (77)) can now be written as:

$$Q(T) = \int_0^T \left(\frac{\partial H}{\partial \theta}\right)^T \left(\frac{\partial H}{\partial \theta}\right) / \psi^2(t)dt \tag{82}$$

3.5. Optimum Sensor Location (OSL) for Conditional estimation ofVector-Valued Parameters

3.5.1. Some Motivative Thoughts and the Fisher Information Matrix (F. I.)

Consider a case in which one tries to estimate one parameter, θ_1, which is to be identified in a dynamic system model with only one sensor provided. One would want to ideally choose a location i (out of N possible such locations) such that the measurement $y_i(t)$, $i \in [1, N]$, $t \in (0, T)$ at location "i" yields the "best" estimate of the parameter θ_1. Heuristically, one should place the sensor at such a location that the time history of the measurements obtained at that location is most

sensitive to any changes in the parameter θ_1. Hence, in equation (75) it is really the slope of $H[X(\theta_1, t)]$ with respect to θ_1 that needs to be maximized. However, since only the absolute magnitude of this slope is of interest, it is logical to want to find i (or equivalently determine the selection matrix S described previously) such as to maximize $(\partial H/\partial\theta_1)^2$. Since this quantity is a function of time one would want to locate a sensor which maximizes the "average" value of $(\partial H/\partial\theta_1)^2$ over the time interval (0, T) during which the response is to be measured. This leads to maximizing the following integral:

$$q_i(T) = \int_0^T \left(\frac{\partial H}{\partial \theta_1}\right)^2 dt. \tag{83}$$

When there is more than one parameter to be estimated, and the number of sensors is greater than unity, this intuitive approach needs to be extended in a more rigorous manner. In such cases recourse to mathematical treatment is necessary, and we shall see that such treatment will be in agreement with our heuristic solution.

3.5.2. Fisher Information Matrix in terms of the measurement response for vector-valued problem

It is shown that for any conditionally unbiased estimator of θ the covariance estimation error can be written as

$$E[(\theta - \hat{\theta})(\theta - \hat{\theta})^T] \geq \left[\int_0^T \left(\frac{\partial H}{\partial \theta}\right)^T \left(\frac{\partial H}{\partial \theta}\right) / \psi^2(t)dt \right]^{-1} . \tag{84}$$

The integral on the right hand side of the inequality (84) is the well known Fisher Information matrix, as defined previously in equation (82). In order to reduce the error in the estimates, one would equivalently want to maximize a suitable norm (e. g., Trace, etc.) [16] of the Fisher Information matrix Q(T). Therefore, introducing equation (76) into equation (82), (this constitutes an extension of the equation (83) which we heuristically derived for the scalar case, to the vector situation), one obtains:

$$Q(T) = \int_0^T \frac{X_\theta^T R^T P R X_\theta}{\psi^2(t)} dt, \tag{85}$$

where the ij element of X_θ can be written as:

$$[X_\theta]_{ij} = \frac{\partial x_i}{\partial \theta_j} , \; i \, \varepsilon \, [1,N], \; j \; \varepsilon \; (1,m) \tag{86}$$

where $X = \{x_i\}_N$ and $\theta = \{\theta_i\}_L$. We note that the Fisher Matrix is symmetric and is dependent on the length of the record available, as well as the locations of the sensors as determined by the matrix P.

If the m locations where the sensors are to be placed are denoted by s_k, k = 1, 2,..., m, then

$$P = \sum_{k=1}^{m} I_{s_k} \tag{87}$$

where the (N x N) diagonal matrix I_{sk} has all its elements equal to zero except the element of the s_k row, which is unity. Noting that P is a diagonal matrix, equation (85) can be simplified to yield

$$Q[T;s_1,s_2,\dots,s_m;\ S,\theta;I] = \sum_{k=1}^{m} \int_0^T \frac{X_\theta^T r_{s_k}^T r_{s_k} X_\theta \, dt}{\psi^2(t)} \tag{88}$$

where r_{sk} is the s_k row of the matrix R. Also in eq. (87) explicit mention is made of the dependence of the Fisher Matrix on the time length T of the available data, the system S, the parameter vector θ, and the time-variant input I. If the matrix R is diagonal, with diagonal elements ρ_1, ρ_1, ρ_N, then the ij element of the matrix Q, after some manipulation, reduces to

$$Q_{ij}[T;s_1,s_2,\dots,s_m;S,\theta;I] = \sum_{k=1}^{m} \int_0^T \left[\frac{\partial x_{s_k}}{\partial \theta_i} \frac{\partial x_{s_k}}{\partial \theta_j} \left(\frac{\rho(t)_{s_k}}{\psi(t)} \right)^2 \right] dt \tag{89}$$

One notes that each element of Q_{ij} represents the cross-sensitivity of measurement with respect to the response x_{sk} of node s_k.

The optimal sensor locations are then obtained by picking m locations s_k, k = 1, 2,...,m, out

of a possible N, so that a suitable norm of the matrix Q is maximized (e, g., the trace norm, etc...), [13, 16]. This may be specified by the condition

$$\max_{s_k \varepsilon (1,N)} \quad \|Q[T;s_1,s_2,...,s_m;S,\theta;\Gamma]\| \ . \tag{90}$$

3.5.3. Linear and Nonlinear Systems

It is important to note that while nothing was said about the nature of the governing differential equations describing the system models, the methodology presented up to now is quite valid for systems with non-linear differential equations. It should also be noted that the criterion developed by equation (89) does not hinge upon the linearity of the system. The only equations involved are the measurement equation (76) and the relation (84). Therefore, the methodology introduced herein may be applied to the systems governed by non-linear differential equations.

3.6. Choice of Matrix Norms

As mentioned above, Q is a matrix and therefore it is necessary to use a suitable scalar norm of it, $\|Q\|$, to obtain an idea of the information content, about a parameter vector θ, available from sensors at one or more locations, given the input, I(t).

Various norms may be used as scalar measures of performance. Some commonly used norms are[19]:

(1) D - optimality: Minimize the determinant of Q^{-1} or eqivalently maximize the determinant

of Q

(2) A - Optimality: Minimize the trace of Q^{-1}

(3) T - Optimality: Maxinize the trace of Q.

An important advantage of D - optimality is its invariance under scale changes in the parameters and linear transformations of the output. However T - optimality has the advantage that the trace operator is linear and therefore Trace(Q) can be expressed as

$$\text{Trace}\{\, Q(T) \,\} = \sum_{k=1}^{k=m} \bar{q}_{s_k}(T) \tag{91}$$

where

$$\bar{q}_{s_k}(T) = \text{Trace} \left\{ \int_0^T \left[\frac{\partial x_{s_k}}{\partial \theta_i} \frac{\partial x_{s_k}}{\partial \theta_j} \left(\frac{\rho(t)_{s_k}}{\psi(t)} \right)^2 \right] dt \right\} \tag{92}$$

When m is large this relationship allows the optimal sensor locations, got by maximizing Trace(Q) (as given by (90), to be obtained in a simple sequential manner. The algorithm to be used can be described in the following three steps.

Step (1): For each s_k, k = 1,2, . . , N, determine $\bar{q}_{s_k}$;

Step (2): Sort the N numbers $\bar{q}_{s_k}$, k = 1,2, . . ., N, in an array of descending order starting with the largest ;

Step (3): The s_k, k = 1, 2, ...,m locations that correspond to the largest m values of $\bar{q}_{s_k}$ are the m optimal sensor locations .

Should r sensors be already fixed in place at locations s_k, k = 1, 2,..r, the best locations for an additional m sensors can be found by including a further step in the above alogrithm. After performing **Step (2)** perform the following two steps.

Step (2'): From the sorted array obtained in **Step (2)** above, delete $\bar{q}_{s_k}$, k=1,2,..,r ;

Step (3): The $\bar{q}_{s_k}$, k=1,2,. . ., m locations corresponding to the largest values in the remaining sorted array yield the optimal sensor locations for the next m sensors.

Due to computational ease and efficiency with which the Trace criterion can be used, and the simplicity with which the maximization defined in (90) can be carried out, in this sequel we shall exclusively use the Trace criterion. Comparison of the results between A, T and D - optimality will be left for a future study.

3.6.1. Analytical Interpretation of the Trace Norm

In this section we present a more formal interpretation of the Trace norm. Let us introduce an error criterion

$$J = E_{\theta,Y}[f(\theta, \hat{\theta})], \tag{93}$$

then

$$J \approx E_{\theta}\left\{E_{Y|\theta}\left[f(\theta,\theta) + \frac{\partial f}{\partial \hat{\theta}}(\hat{\theta} - \theta) + \frac{1}{2}(\hat{\theta} - \theta)^T \frac{\partial^2 f}{\partial \hat{\theta}^2}(\hat{\theta} - \theta\right]\right\}, \tag{94}$$

where θ, $\hat{\theta}$ and Y are the true value of the estimated parameters, the estimate, and the measurement yielding the estimates, respectively. Since $f(\theta, \hat{\theta})$ is a function of error between the θ and $\hat{\theta}$, then $f(\theta, \theta) = 0$. Hence,

$$J \approx E_{\theta}\left\{E_{Y|\theta}\left[\frac{\partial f}{\partial \hat{\theta}}(\hat{\theta} - \theta)\right] + E_{Y|\theta}\left[\frac{1}{2}(\hat{\theta} - \theta)^T \frac{\partial^2 f}{\partial \hat{\theta}^2}(\hat{\theta} - \theta)\right]\right\}. \tag{95}$$

However, if $\hat{\theta}$ is close to θ, then $\frac{\partial f}{\partial \hat{\theta}} \approx \frac{\partial f}{\partial \theta}$. Using this approximation one can write

$$J = E_{\theta}\left\{\frac{\partial f}{\partial \theta} E_{Y|\theta}(\hat{\theta} - \theta) + E_{Y|\theta}\left[\frac{1}{2}(\hat{\theta} - \theta)^T \frac{\partial^2 f}{\partial \hat{\theta}^2}(\hat{\theta} - \theta)\right]\right\}. \tag{98}$$

Notice that if $\hat{\theta}$ is an efficient unbiased estimator, then $E(\hat{\theta}) = \theta$. Hence,

$$E_{Y|\theta}(\hat{\theta} - \theta) = E_{Y|\theta}(\hat{\theta}) - E_{Y|\theta}(\theta) = 0. \qquad (97)$$

Consequently one can further simplify J. Therefore,

$$J = E_{\theta} E_{Y|\theta}\left[\frac{1}{2}(\hat{\theta} - \theta)^T \frac{\partial^2 f}{\partial \hat{\theta}^2}(\hat{\theta} - \theta)\right]. \qquad (98)$$

Noting that

$$\mathrm{Cov}\,\hat{\theta} = E_{Y|\theta}[(\hat{\theta} - \theta)(\theta - \hat{\theta})^T \mid \theta], \qquad (99)$$

after some matrix manipulation one can write

$$J = E_{\theta}\left[\frac{1}{2}\mathrm{Trace}\left(\frac{\partial^2 f}{\partial \hat{\theta}^2}\mathrm{Cov}\,\hat{\theta}\right)\right]. \qquad (100)$$

To minimize the error between the estimate $\hat{\theta}$ and θ, one would want to minimize the right hand side of the above equation. If f is quadratic in θ then, the second derivative of f with respect to θ in (100) is a constant matrix.

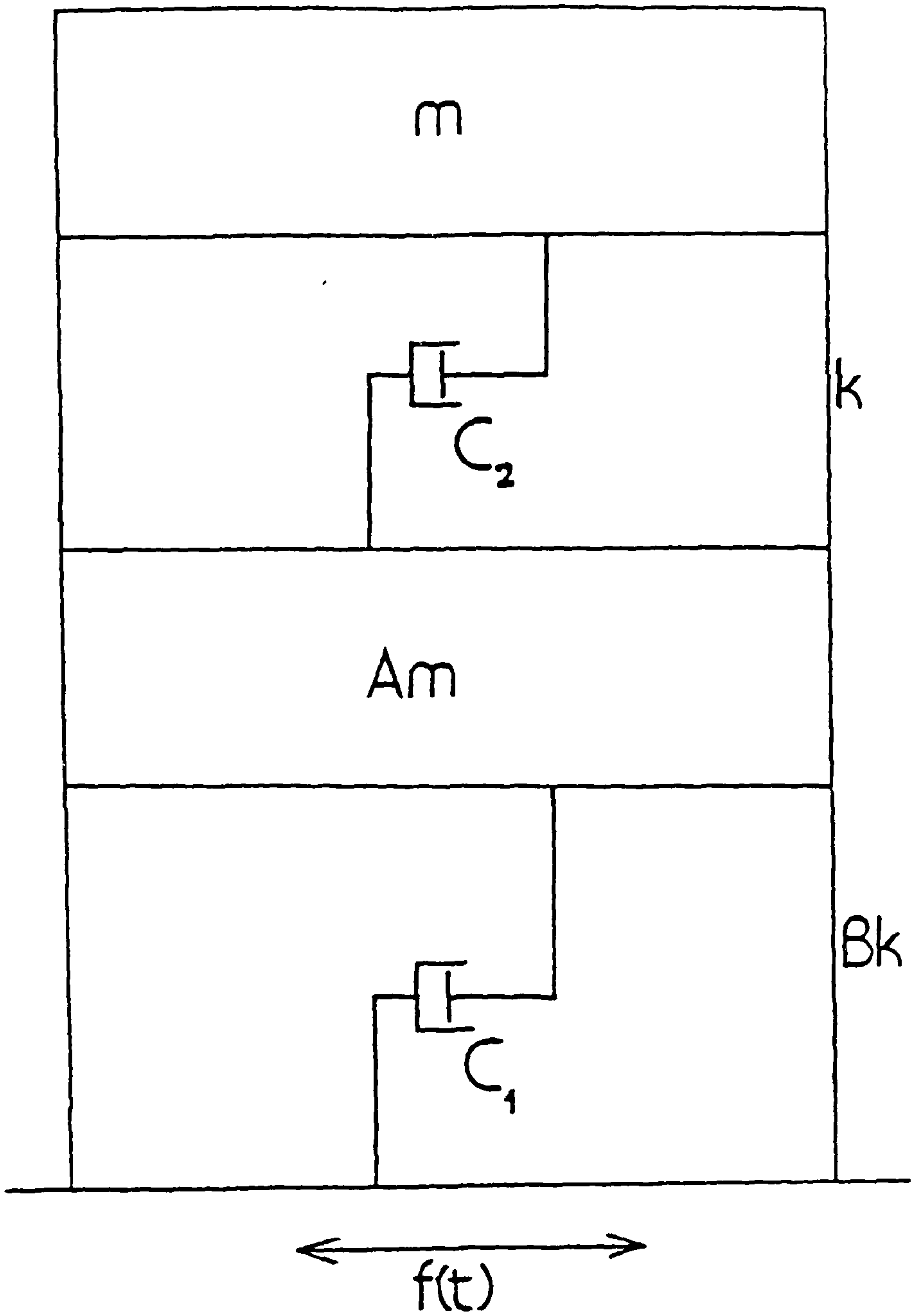


Figure 4. A two Mass System subjected to Support Excitation, f(t)

3.7. Application to a two-degree-of-freedom linear system

Consider the problem of finding the optimal sensor location in a two mass system, modelled by the two-degree-of-freedom of Figure 4, so as to "best" identify: (1) the mass ratio, A, of the first to second mass; (2) the stiffness ratio, B, of the lower to the upper spring; and (3) the damping coefficient.

The governing differential equation of motion can be expressed as

$$M\ddot{X} + C\dot{X} + KX = Wf(t)\ ;\ \dot{X}(0) = 0\ \ , X(0) = 0\ , \tag{101}$$

where $X = \langle x_1\ x_2\rangle^T$, $C = \alpha K$, $W = \langle Am\ m\rangle^T$ and f(t) is the base excitation. The matrices M and K are

$$M = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} m \quad , \quad \text{and} \quad K = \begin{bmatrix} B+1 & -1 \\ -1 & +1 \end{bmatrix} k\ . \tag{102}$$

Let s_1 denotes the lower mass location and s_2 the upper mass location. The selection between the locations can be equated to determining the one non-zero element of selection matrix, S $_{[1x2]}$, with the measurement H(t) defined by H(t) = SX + V(t), where V(t) is assumed to be stationary Gaussian White Noise (SGWN) with $\psi(t) = \psi_0$. As x_1 denotes the lower mass response, if S = [1 0} the lower mass is selected for measurement; if S = [0 1] the upper mass is selected. The location s_1 would then be preferred over the location s_2 for identifying the parameter

A (for instance), if $Q[T,s_1] > Q[T, s_2]$, where T is the time that the measurement is taken,

$$Q_1(T) \triangleq Q_1(T, s_1) = \frac{1}{\psi_0^2} \int_0^T \left(\frac{\partial x_1}{\partial A} \; \frac{\partial x_2}{\partial A} \right) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial x_1}{\partial A} \\ \frac{\partial x_2}{\partial A} \end{Bmatrix} dt$$

$$= \frac{1}{\psi_0^2} \int_0^T \left(\frac{\partial x_1}{\partial A} \right)^2 dt \tag{103}$$

and,

$$Q_2(T) \triangleq Q\ (T, s_2) = \frac{1}{\psi_0^2} \int_0^T \left(\frac{\partial x_1}{\partial A} \; \frac{\partial x_2}{\partial A} \right) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \frac{\partial x_1}{\partial A} \\ \frac{\partial x_2}{\partial A} \end{Bmatrix} dt$$

$$= \frac{1}{\psi_0^2} \int_0^T \left(\frac{\partial x_2}{\partial A} \right)^2 dt\,. \tag{104}$$

Since there is only one parameter being estimated (say A), the Fisher Information matrices reduce to scalar quantities.

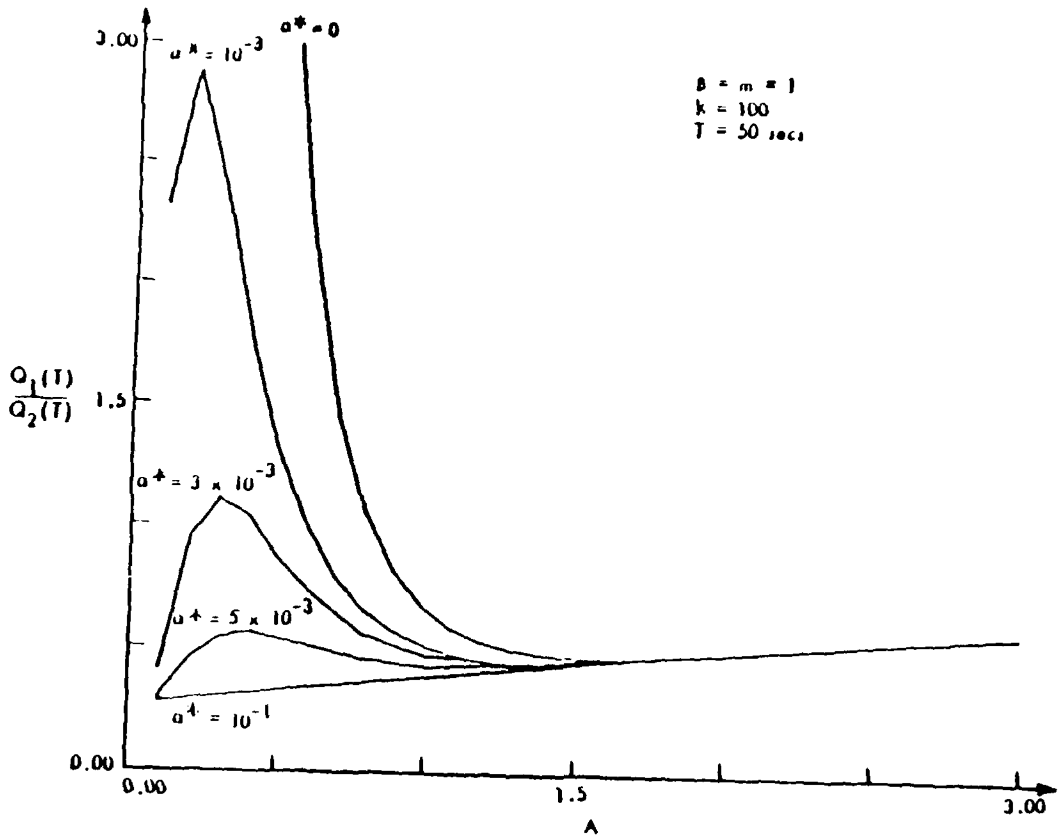


Figure 5A. Variation of $Q_1(T)/Q_2(T)$ with A for different $\alpha^* \triangleq \alpha\omega_o$ given B = m = 1, K = 100, and $f(t) = \delta(t)$.

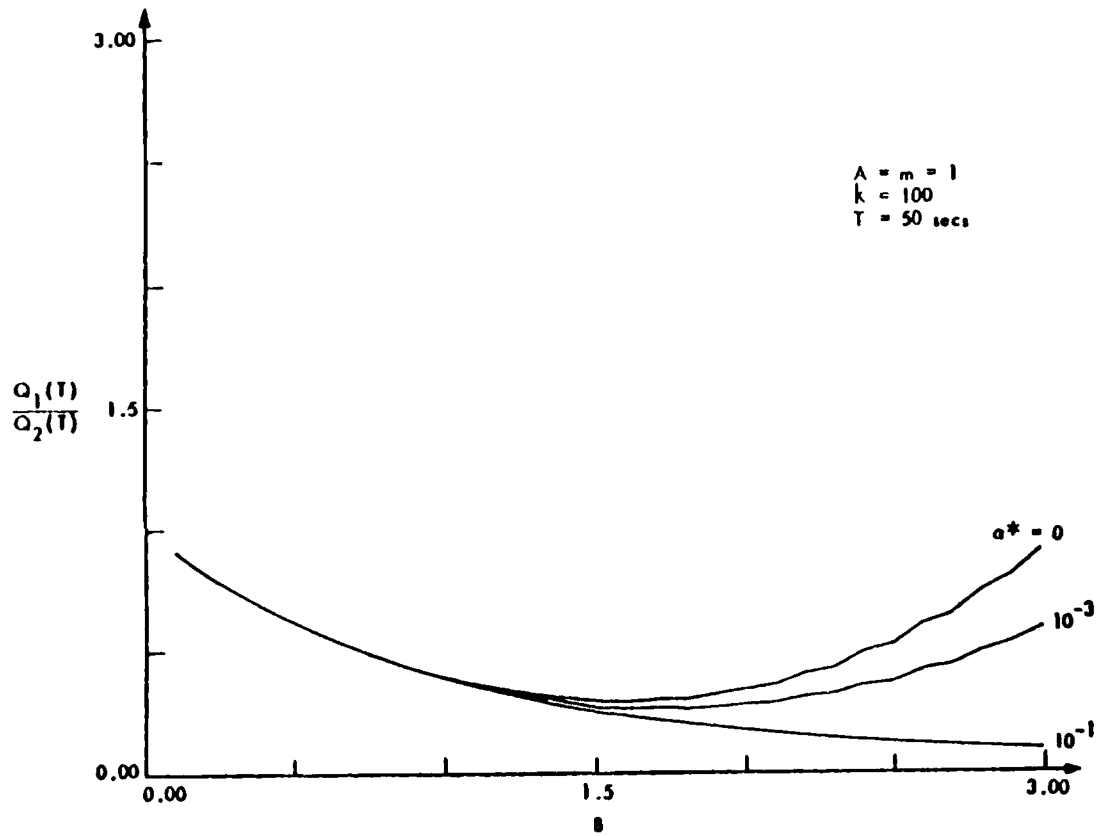


Figure 5B. Variation of $Q_1(t)/Q_2(t)$ with B for different $\alpha^* \triangleq \alpha\omega_0$ given $A = m = 1$, $k = 100$, and $f(t) = \delta(t)$.

The dependence of the OSL on various type of the base excitations has been studied by considering various base accelerations. *In all the examples that follow it will be asumed that the parameter values corresponding to the M,K, and C matrices (of equation (69)) are provided in appropriate and consistent units.*

3.7.1. Example of the 2-degree-of-freedom-system (2-DOFS) subjected to impulsive base excitation

Consider the OSL problem for the "best" (minimum covariance) identification of the parameter A (given the parameter B and α) of the two-mass system governed by the differential equation (28) using an impulsive base excitation ($f(t) = \delta(t)$). Figure 5-A shows the plots of the ratio of the information matrices $Q_1(T)/Q_2(T)$, for T = 50 seconds, for various values of the parameters A (which is to be identified) and, $\alpha^* = \alpha\omega_0$, where $\omega_0 = \sqrt{k/m}$. Points on the graph with ordinates greater than unity indicates the optimal location to be the lower mass level and vice-versa. The closed form solution for the undamped case is provided in the Appendix A. Figure 5-A indicates that the optimal location in most cases, for the range of A considered, is the upper mass level. However, we observe that for some small values of A and α^* the OSL is the lower level. We note, interestingly enough, that the optimal sensor location for identification of A actually depends not only on the value of B and α^* which are presumably known, but also on the value of the parameter A which **itself is to be identified!** Thus to be able to ascertain the optimal sensor location some apriori assessment of A is necessary. This brings us back to our discussion of section 3.2 where we emphasized that the optimal sensor location problem is looked

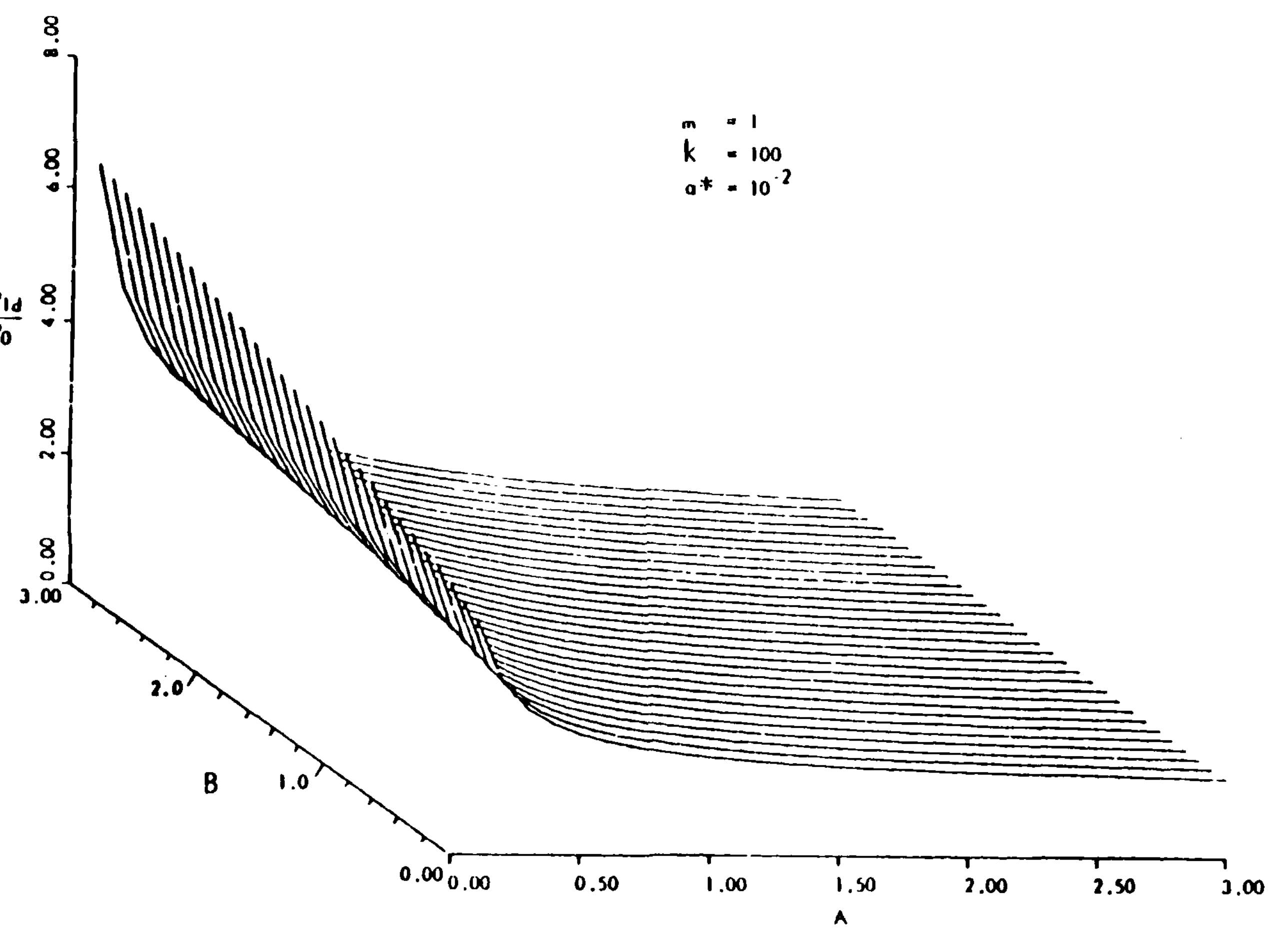


Figure 6A. Variation of the damped natrual frequencies (normalized with respect to ω_o) with A and B for m=1, k=100 and $\alpha^*=10^{-2}$.

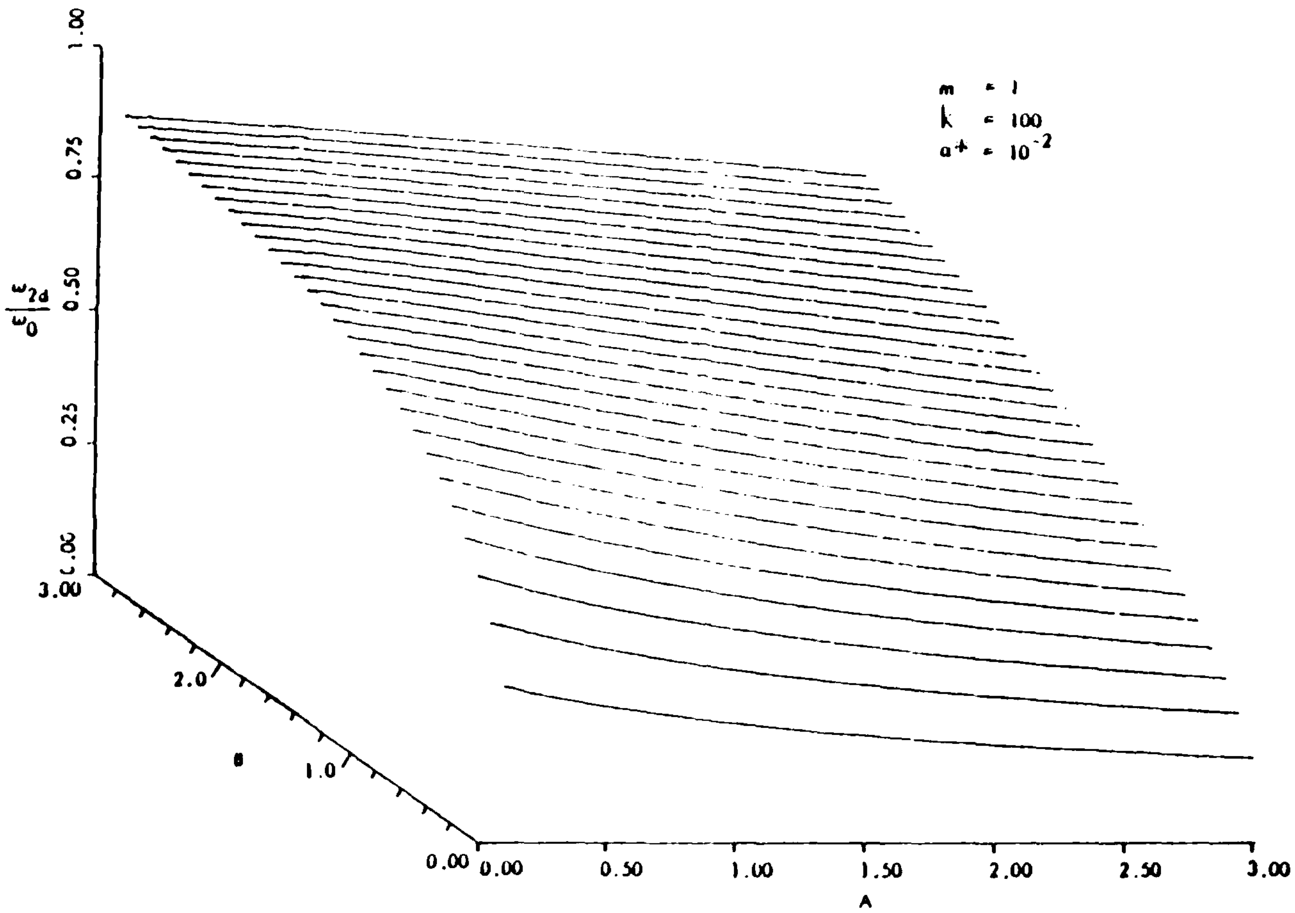


Figure 6B. Variation of the damped natural fequencies (normalized with respect to ω_0) with A and B for m=1, k=100 and $\alpha^*=10^{-2}$

at herein in the "local" context.

Consider next the OSL for the "best" identification of parameter B (given A and α). The exact solution of this problem is also provided in Appendix A. Figure 5-B shows that the optimal location for identification of parameter B, using an impulsive base input, is again the upper mass level for the range of B, however, and $\alpha^* > 0.05$ ($\alpha^* = \alpha\omega_0$, $\omega_0 = \sqrt{k/m}$) the trend appears to be more and more in favor of the upper mass. This seems intuitively correct for as B becomes larger, the lower mass becomes immobile and the OSL would be the upper mass.

3.7.2. Example of the Two degrees-of-freedom-system (2-DOFS) subjected to sinusoidal base excitation

Consider the OSL for "best" identification of parameter A, B, and α of the two-mass structure governed by the differential equation (101) when $f(t) = A\ \mathrm{Sin}(\omega t)$.

Figure 6-A and 6-B show for convenience how the system damped natural frequencies, normalized with respect to ω_0 ($\omega_0 = \sqrt{k/m}$) , change with changes in A and B. In Figure 6-A, the effect of A values in the vicinity of $A < 1$ on the natural frequencies is highly noticeable. In Figure 6-B, however, the frequency curves become less sensitive to changes in B for large enough values of B.

Figure 7-A is associated with the solution of the OSLP for estimating the parameter A or B using sinusoidal base excitation. This figure shows that as the normali:ed driving frequency γ (γ =

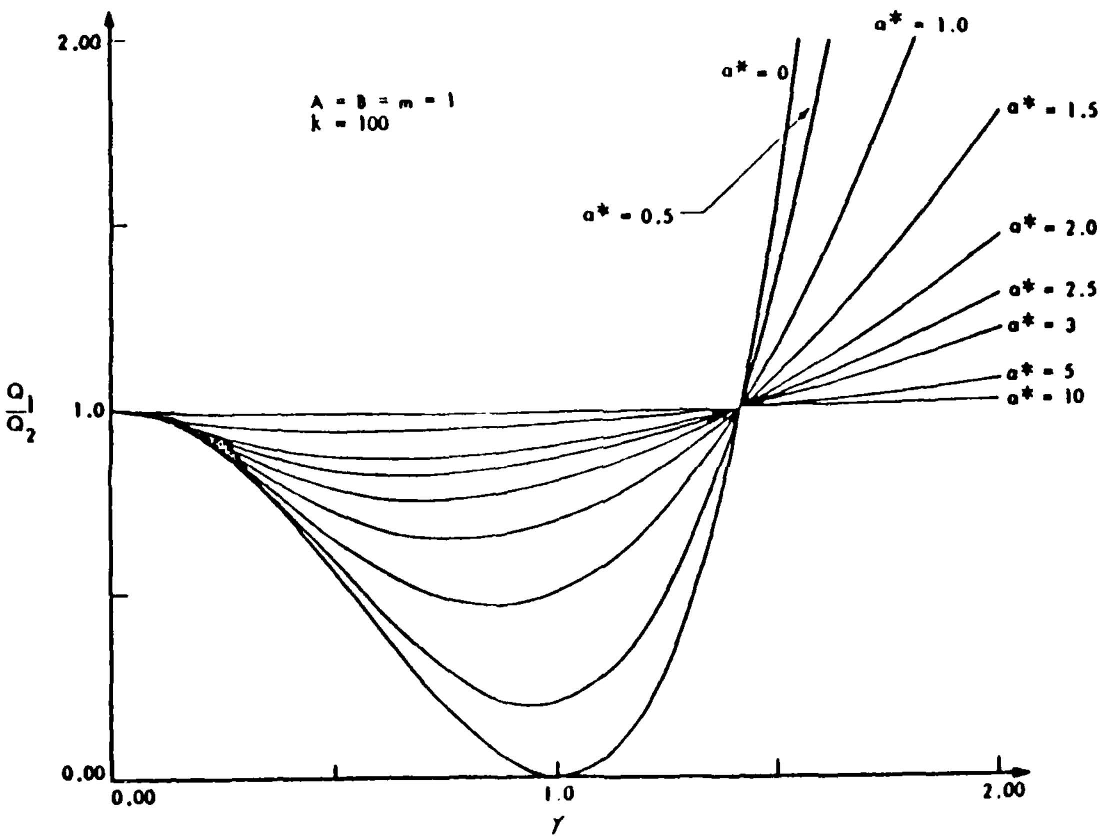


Figure 7A. Variation of Q_1/Q_2 with $\gamma \triangleq \omega/\omega_0$ for different α^* given A=B=m=1, k=100 and f(t)=Sinωt

ω/ω_0) varies, the OSL changes. The exact forms of the Fisher Information matrices are expressed in Appendix B. We observe that for the case when $\alpha^* = 0$,

$$\frac{Q_1}{Q_2} = (1 - \gamma^2)^2 , \tag{105}$$

when estimating the parameter A (or B). Figure 7-A indicates that the OSL is always the upper mass level for $\gamma < \sqrt{2}$ and the lower mass for $\gamma > \sqrt{2}$. The dependence of the Fisher value ratio α^* is also illustrated. It is of some significance to also note that the information at both floors vanish (for estimating A) when $\alpha^* = 0$ as $\gamma \rightarrow 1$. Thus the region in frequency space around $\gamma = 1$ is not a good region for exciting the base of the structure sinusoidally if the aim is to obtain records for the identification of A. The information from records available at either level is however of equal value (so far as estimating A is concerned) for $\gamma = 0$ and $\gamma = \sqrt{2}$.

For the estimation of B (given A and $\alpha^* = 0$) the dimensionless driving frequency $\gamma = \sqrt{1 + \frac{1}{A}}$ (derived in Appendix B) yields no information on B from records at either of the two mass levels. On the other hand, the responses at the two mass levels yields identical amounts of information on B at $\gamma = 0$ and $\gamma = \sqrt{2}$ (for $A \neq 1$), as indicated by the value of $Q_1/Q_2 = 1$ at these frequencies.

The value of the $Q_1/Q_2 = 0$ at $\gamma = 1$ is indicative of the fact that the upper mass level is a far better location for a sensor (when estimating B) and $\alpha^* = 0$.

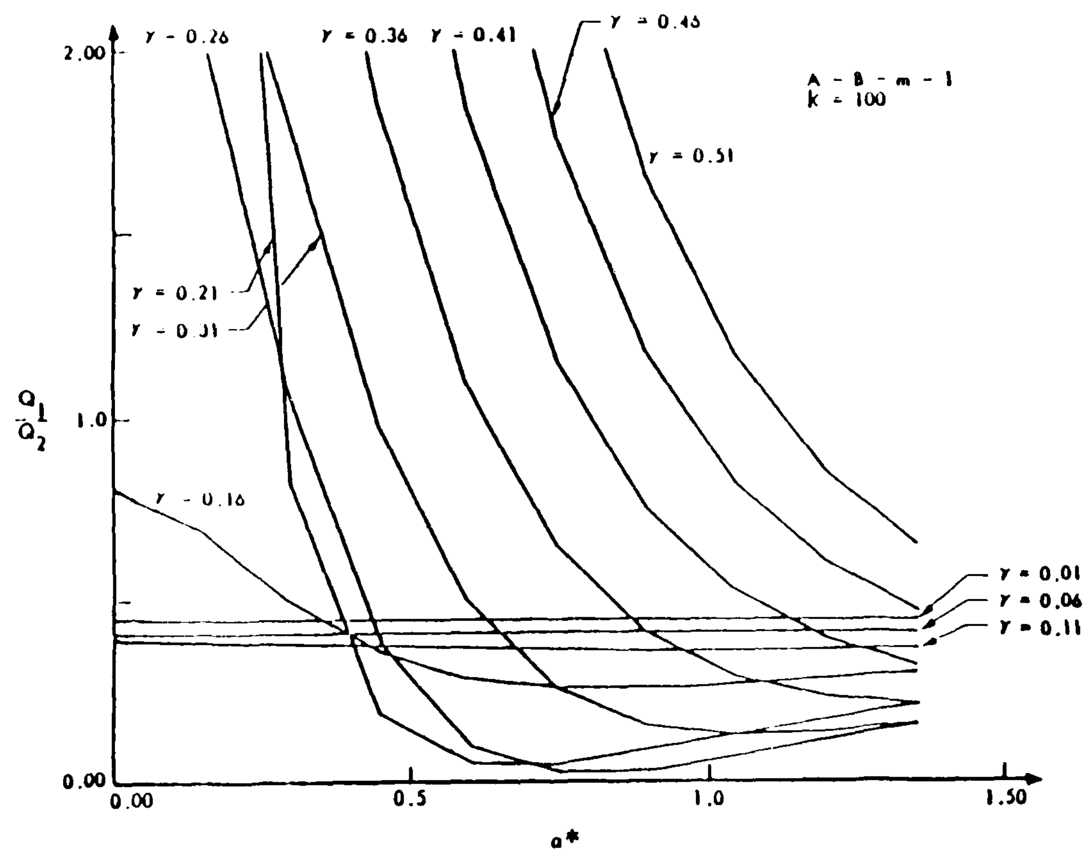


Figure 7B. Variation of Q_1/Q_2 with α^* for various values of γ, given A=B=m=1, k=100 and f(t) =Sinωt

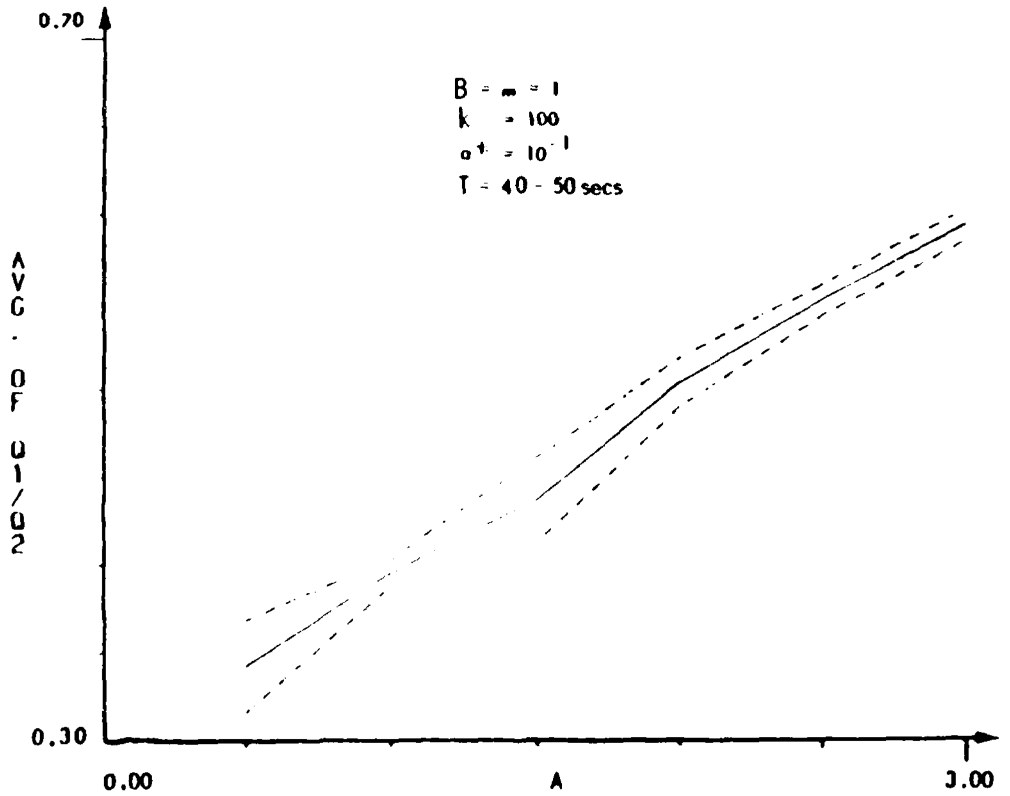


Figure 8A. Variation in the mean value of $Q_1(T)/Q_2(T)$ and the $1\text{-}\sigma$ band with different values of A, when B=m=1, k=100, $\alpha^*=0.1$. Base input is Gaussian White Noise. Integration was done over a ten second period

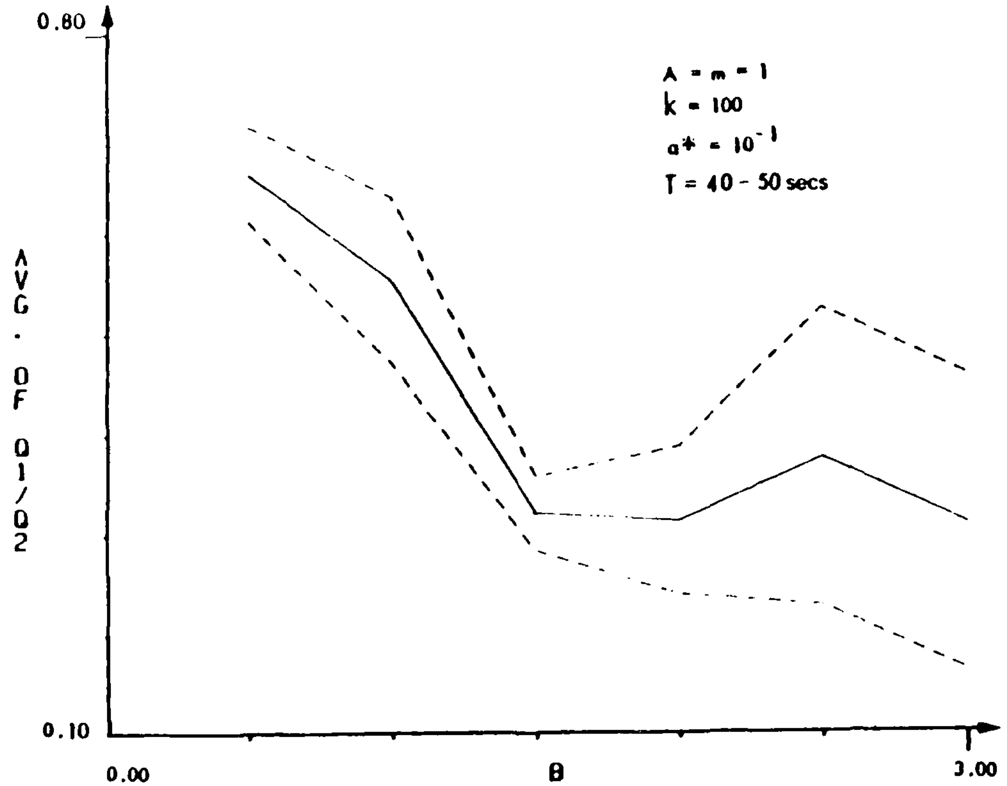


Figure 8B. Variation in the mean value of $Q_1(T)/Q_2(T)$ and the 1-σ band with changes in B when A=m=1, k=100, α*=0.1 for Gaussian White Noise input

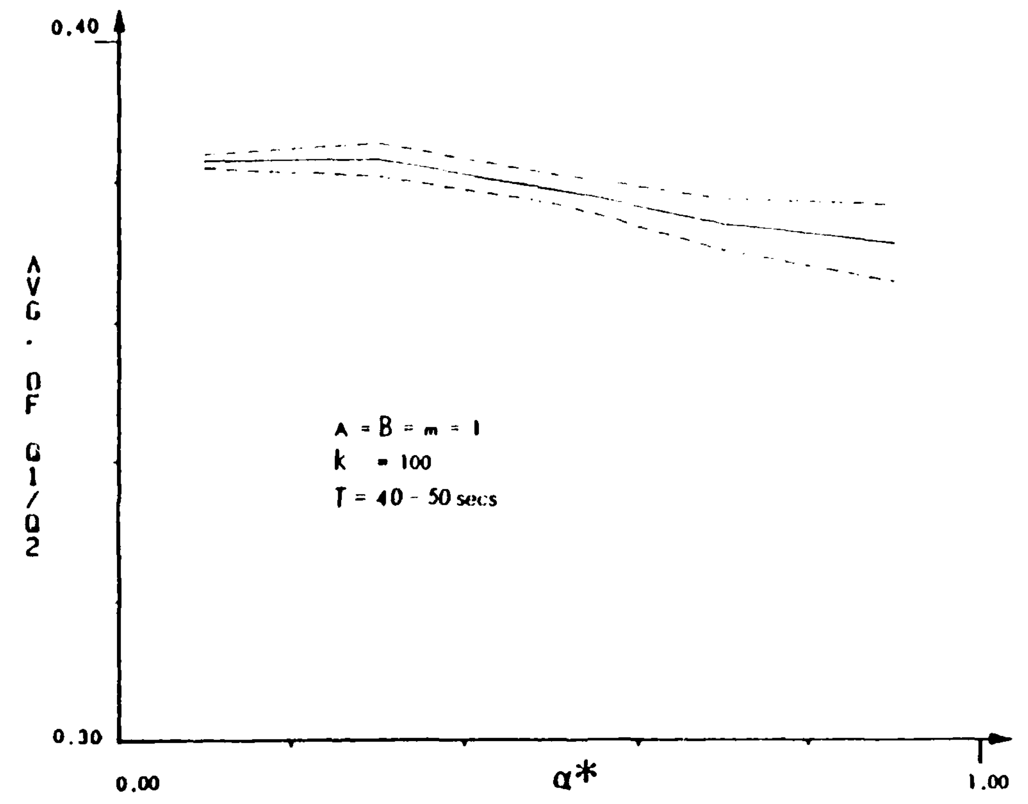


Figure 8C. Variation in the mean value of $Q_1(T)/Q_2(T)$ and the $1-\sigma$ band when A=B=m=1 and k=100 for Gaussian White Noise input

Figure 7-B indicates the OSL solution for the situation wherein α^* ($\alpha^* = \alpha\omega_0$) is to be estimated. The figure indicates the dependence of the OSL on the driving frequency γ and the normalized damping value, α^*, for the system. For the particular system chosen, for example, when $\gamma \leq 0.16$, the OSL is the upper mass level. For values of $\gamma > 0.16$, the OSL depends on the value of the critical damping $\zeta_n = a\,\omega_n/2$, where ω_n is the natural frequency of the system (n = 1, 2). As the damping increases, the OSL moves towards the upper mass.

3.7.3. Example of the 2-degree-of-freedom-system (2-DOFS) subjected to zero mean Gaussian white noise (ZMGWN) base excitation

Consider the same two mass structure governed by differential equation (101). Let f(t) be taken as a sample of Gaussian White Noise (GWN). Figures 8-A, 8-B, and 8-C are associated with the OSLP for estimating, A, B, and α^* ($\alpha^* = \alpha\omega_0$) respectively. For each case five different realizations of a GWN process were used. The expected values of Q_1/Q_2 are indicated in each plot with a solid line. The dashed lines indicate the one-standard deviation, (1-σ), band in which the ratio of the Fisher's lie. The results indicate that for damped structural systems ($\alpha^* > 0$), the OSL in most cases is the upper mass level. Hence, one would want to locate the sensor on the top mass to obtain records yielding estimation of A, B and α^*.

3.8. Optimal Sensor Locations for N-Degree-of-Freedom Linear Systems

In this sequel the solution to the OSLP will be generated for multi-degree-of-freedom linear systems. Consider the N-degree-of-freedom, classically damped, dynamic system whose

governing differential equation of motion can be written as :

$$M\ddot{X} + C\dot{X} + KX = F(t), \ X(t_0) = X_0, \ \dot{X}(t_0) = \dot{X}_0, \tag{106}$$

where $\dot{X}_0$ and X_0 are the given initial conditions for the system. Using normal mode method the response vector X(t) can be determined. Introducing

$$X(t) = \Phi\eta(t) \tag{107}$$

where Φ is the (N x N) weighted model matrix (transformation matrix) and $\eta(t)$ may be refered to as the N-vector of generalized coordinates (response coordinates). Then,

$$\ddot{\eta}+2\xi_N\omega_N\dot{\eta}+\Lambda\eta = \Phi^T F(t), \quad \eta(t_0) = \phi^T M X_0, \ \dot{\eta}(t_0) = \phi^T M\dot{X}_0, \tag{108}$$

where the (N x N) diagonal matrix $\mathcal{L}$ (sometimes known as the generalized stiffness matrix) is

$$[\ \Lambda\ ] = \Phi^T K \Phi = \mathrm{Diag}[\ \omega_1, \omega_2, . . , \omega_N] \tag{109}$$

The solution of equation (107) is given as

$$\eta_i(t) = \eta_{0_i} u_i(t-t_0)+\dot{\eta}_{0_i}\ v_i(t - t_0) + \int_{t_0}^{t} h_i\,(t-\tau)p_i\,(\tau)d\tau \tag{110}$$

where η_{0i} and $\dot{\eta}_{0i}$ arė the initial conditions and,

$$u_i(t) = \mathrm{EXP}\,(-\xi_i\omega_i t)\left[\mathrm{Cos}\omega_{d_i} t + \frac{\xi\omega_i}{\omega_{d_i}}\,\mathrm{Sin}\omega_{d_i} t\right], \tag{111}$$

$$v_i(t) = \frac{1}{\omega_{d_i}}\mathrm{EXP}(-\xi_i\omega_i t)\mathrm{Sin}\omega_{d_i} t, \tag{112}$$

$$h_i(t) = v_i(t), \tag{113}$$

$$\omega_{d_i} = \omega_i\sqrt{1 - \xi_i^2}\,, \quad \text{and} \tag{114}$$

$$p_i(t) = \Phi^T F(t), \quad i = 1,2,\ldots,N. \tag{115}$$

If the OSLP is to be solved for estimation of the parameter vector Θ, where Θ was previously defined in eq. (70), one should differentiate equation (105) with respect to Θ. This yields:

$$M\ddot{X}_\Theta + C\dot{X}_\Theta + KX_\Theta = F_\Theta(t) - (\overline{M_\Theta\ddot{X}}+\overline{C_\Theta\dot{X}}+\overline{K_\Theta X});\; X_\Theta(t_0) = 0,\; \dot{X}_\Theta(t_0) = 0 \tag{116}$$

where

$$[X_\Theta]_{ij} = \frac{\partial x_i}{\partial \theta_j}\,, \tag{117}$$

$$[\overline{M_\Theta \ddot{X}}] = \{\, M_{\theta_1}\ddot{X} \mid M_{\theta_2}\ddot{X} \mid \ldots \mid M_{\theta_L}\ddot{X} \,\} \tag{118}$$

$$\overline{[C_\Theta \dot{X}]} = \{ C_{\theta_1} \dot{X} \mid C_{\theta_2} \dot{X} \mid . . \mid C_{\theta_L} \dot{X} \} \tag{119}$$

$$\overline{[K_\Theta X]} = \{ K_{\theta_1} X \mid K_{\theta_2} X \mid . . \mid K_{\theta_L} X \} \tag{120}$$

$$[F_\Theta(t)]_{ij} = \frac{\partial f_i}{\partial \theta_j} \quad ; \qquad \text{with} \tag{121}$$

$$\Theta = \langle \Theta_M \vdots \Theta_C \vdots \Theta_K \rangle_L^T = \langle \theta_j \rangle_L^T \tag{122}$$

for $\quad i = 1,...,N$, and $j = 1,...,L$.

Introducing

$$X_\Theta = \Phi z(t) \tag{123}$$

yields

$$\ddot{z} + 2\xi_N \omega_N \dot{z} + \Lambda z = G(t); \ \dot{z}(t_0) = 0 \ ; \ z(t_0) = 0 \tag{124}$$

where

$$G(t) = \Phi^T [F_\Theta - (\overline{M_\Theta \ddot{X}} + \overline{C_\Theta \dot{X}} + \overline{K_\Theta X})] \ . \tag{125}$$

Equation (125) can further be simplified using (123) to

$$G(t) = \Phi^T[F_\Theta - (\overline{M_\Theta \Phi \ddot{\eta}} + \overline{C_\Theta \Phi \dot{\eta}} + \overline{K_\Theta \Phi \eta})\ ] \tag{126}$$

where $\dot{\eta}$ and $\ddot{\eta}$ can be obtained by proper differentiation of eq. (109). This may be shown as follow

$$\dot{\eta}_i(t) = \eta_{0_i} W_i(t-t_0) + \dot{\eta}_{0_i} Y_i(t - t_0) + \int_{t_0}^{t} \overline{h}_i(t - \tau) p_i(\tau) d\tau \tag{127}$$

where

$$W_i(t) = -\ \text{EXP}(-\xi_i \omega_i t) \left[\omega_{d_i} + \frac{(\xi_i \omega_i)^2}{\omega_{d_i}} \right] \text{Sin}\omega_{d_i} t\,, \tag{128}$$

$$Y_i(t) = \text{EXP}(-\xi_i \omega_i t) \left[\text{Cos}\omega_{d_i} t - \left(\frac{\xi_i \omega_i}{\omega_{d_i}} \right) \text{Sin}\omega_{d_i} t \right], \tag{129}$$

$$\overline{h}_i(t) = Y_i(t), \quad \text{and} \tag{130}$$

$$p_i(t) = \Phi^T F(t)\,, \quad i = 1,2,\ldots,N\,. \tag{131}$$

Also

$$\ddot{\eta}_i(t) = \eta_{0_i} \overline{W}_i(t - t_0) + \dot{\eta}_{0_i} \overline{Y}_i(t - t_0) + \int_{t_0}^{t} \overline{\overline{h}}_i(t - \tau) p_i(\tau) d\tau \tag{132}$$

where

$$\overline{W}_i(t) = EXP(-\xi_i\omega_i t)\left\{\left[\frac{(\xi_i\omega_i)^3}{\omega_{d_i}} + \omega_{d_i}(\xi_i\omega_i)\right]Sin\omega_{d_i}t \right. \tag{133}$$

$$\left. - [\omega_{d_i}^2 + (\xi_i\omega_i)^2]\, Cos\omega_{d_i}t \right\} ,$$

$$\overline{Y}_i(t) = EXP(-\xi_i\omega_i t)\left\{\left[\frac{(\xi_i\omega_i)^2}{\omega_i} - \omega_{d_i}\right]Sin\omega_{d_i}t \right. \tag{134}$$

$$\left. - 2\xi_i\omega_i\, Cos\omega_{d_i}t \right\} ,$$

$$\overline{\overline{h}}_i(t) = \overline{Y}_i(t) \tag{135}$$

$$p_i(t) = \Phi^T F(t), \quad i = 1,2,...,N. \tag{136}$$

Therefore, substituting equations (127) and (132) into equation (126) gives G(t). Consequently the solution of equation (124) can be written as:

$$z_{ij}(t) = \int_{t_0}^{t} h_i(t-\tau)G_{ij}(\tau)d(\tau) \tag{137}$$

where $h_i(t)$ is the same as in eq. (113). Notice that the initial conditions in eq. (124) are zero. This is due to the fact that the initial conditions of (106) are known constants.

If in the eq. (106) we assume that [C] is expressed as a linear combination of [K] and [M], then eq. (126) can further be simplified. Namely,

$$C = 2\alpha K + 2\beta M \tag{138}$$

where α, and β are known constants. Hence in equation (108), the percentage of damping, ξ_N, can be expressed as:

$$\xi_i = \alpha\omega_i + \frac{\beta}{\omega_i}, \quad i = 1,2,\ldots,N \tag{139}$$

Then eq. (126) may be written as:

$$G(t) = \Phi^T\{F_\Theta - [M_\Theta\overline{\Phi(\ddot{\eta}+2\beta\dot{\eta})} + K_\Theta\overline{\Phi(\eta+2\alpha\dot{\eta})} + 2(\beta_\Theta \overline{M+\alpha_\Theta K)\Phi\dot{\eta}}\} \tag{140}$$

To futher simplify eq. (140), let us consider the following three cases:

(1) If vector Θ contains Θ_M only (estimating masses), then

$$G(t) = \Phi^T[F_\Theta - \overline{M_\Theta\Phi(\ddot{\eta}+2\beta\dot{\eta})}] \ . \tag{141}$$

(2) If, however, vector Θ of eq. (70) only contains the subvector Θ_K (estimating stiffness),

then

$$G(t) = \Phi^T[F_\Theta - \overline{K_\Theta \Phi(\eta + 2\beta\dot{\eta})}] \quad . \tag{142}$$

(3) Finally if vector Θ is reduced to subvector Θ_C , $\Theta_c = [\ \alpha\ \ \beta]^T$, then

$$G(t) = \langle \Phi^T F_\alpha - 2\Lambda\dot{\eta} \ \vdots\ \Theta^T F_\beta - 2I\dot{\eta} \rangle \tag{143}$$

where I is the identity matrix. It should be realized that if the input F(t) is independent of Θ, then F_Θ would be cancelled all through this discussion.

Once the solution of eq. (116) is obtained using (123), the Fisher matrices may be evaluated using eq. (85). Hence,

$$Q = \sum_{k=1}^{m} \int_{t_0}^{T} \frac{z^T \Phi^T r_{s_k}^T r_{s_k} \Phi z}{\psi^2(t)} dt \tag{144}$$

3.8.1. Some useful results

Let us assume that for a given N-degree-of-freedom dynamic system one seeks the solution of the OSLP for the purpose of all the system model parameters $\{\theta\}_L$. Let R= I and $\Psi(t)= \Psi_0$.

With the equation governing the motion expressed (69), following the solution steps presented in the previous section, one can write:

$$X_{\Theta} = [\Phi]_{N\times N}[z_1 z_2 \cdots z_L]_{N\times L} \; . \tag{146}$$

To obtain the Fisher Information Matrix, we shall utilize equations (85) and (88). Hence using eq. (88) one can write:

$$I_{s_K} = \begin{bmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow s_k \text{ - th row,} \\ \\ \\ \\ \\ \end{array} \tag{147}$$

and

$$H_{s_k} \triangleq \Phi^T I_{s_k} \Phi \; . \tag{148}$$

Therefore, equation (144), under our assumptions with $t_0 = 0$, can be written as:

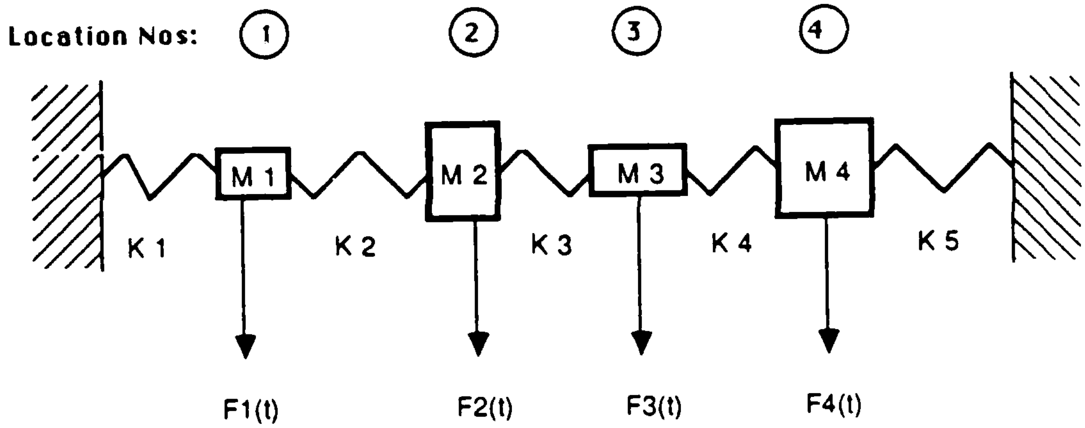
$$Q = \sum_{k=1}^{m} \int_0^T \begin{Bmatrix} z_1^T \\ z_2^T \\ \cdot \\ \cdot \\ \cdot \\ z_L^T \end{Bmatrix} [H_{s_k}] \quad <z_1 z_2 \cdots z_L> dt \tag{149}$$

where m is the number of sensors to be used. The above equation can further be expanded to:

$$Q = \sum_{k=1}^{m} \int_0^T \begin{bmatrix} z_1^T H_{s_k} z_1 & \cdots & z_1^T H_{s_k} z_L \\ z_2^T H_{s_k} z_1 & \cdots & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ z_L^T H_{s_k} z_1 & \cdots & z_L^T H_{s_k} z_L \end{bmatrix} dt \tag{150a}$$

Equation (150) is the Fisher Information matrix for the given N-degree-of-freedom dynamic system. If θ_i is <u>not</u> to be estimated, then the i-th row and the i-th column of the matrix of equation (150) would be absent. Therefore, if θ_1 and θ_3 of Θ were to be estimated, then the first and the third rows and columns in the matrix equation (150) would only be presented. Namely,

$$Q = \sum_{k=1}^{m} \int_0^T \begin{bmatrix} z_1^T H_{s_k} z_1 & z_1^T H_{s_k} z_3 \\ z_3^T H_{s_k} z_1 & z_3^T H_{s_k} z_3 \end{bmatrix} dt . \tag{150b}$$



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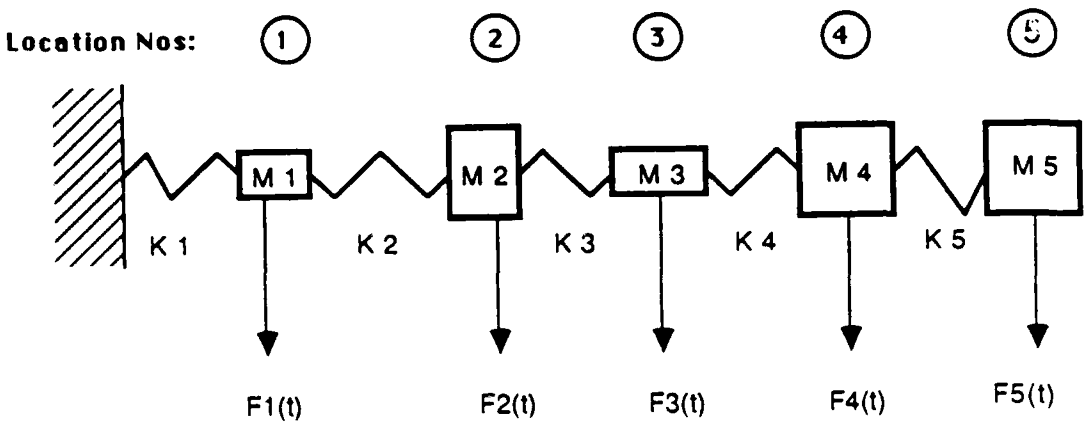
(a)



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(b)

Figure 9. Multi-degree of Freedom Systems for Numerical Studies

This result is useful for computational purposes.

3.9. Applications to Multi-Degree-of Freedom Systems

Figure 9 shows two different multi-degree-of freedom systems that will be used to illustrate the OSLP methodology developed. The numerical results obtained will indicate the nature of the solutions for the OSLP and oftentimes their non-intuitive character. To illustrate the dependence of the OSL on the nature of the location (s) and types of inputs, two types of excitations have been used -- transient and impulsive.

3.9.1. Fixed-Fixed System of Figure 9(A)

Figure 9(A) shows a four degree of freedom system. The system parameters are: m_1=m_2=2; m_3=m_4=1; k_1=k_2=100; k_3=75; k_4=50; k_5=50. The damping is taken to be of Raleigh form (i.e. C= $2\alpha M + 2\beta K$) with α=0.001 and β= 0.04. The measurement noise $\Psi(t)$ is taken to be ψ_0.

Further, it is assumed that we have the ability to apply an impulsive force f(t) (to any one of the masses m_i, i = 1,2,3,4) whose impulse

$$I = \int_0^\infty f(t)\,dt = 10 \qquad (151)$$

The parameter values provided are assumed to be in appropriate and consistent units. We shall investigate the following :

1) If the impulsive force described above is applied to one of the masses, say mass m_j, j ε (1,4), then where should we locate a sensor to best identify one of the stiffnesses k_i, i ε (1,5)?

2) Were we required to place more than one sensor to identify k_i how would we find the optimal locations ? Could we rank order the locations 1,2,3,4 shown in Figure 9(A) indicating the order in which they should be populated by sensors so as to best identify k_i ?

3) Can we get an idea regarding the information gained (or reduced) by placing a sensor at location r as opposed to location k (r,k ε (1,4)) ?

4) Given that we want to identify k_i using an impulsive force which can be applied at one of the masses, at which mass should it be applied and at which locations should the corresponding responses be measured for best identification?

5) What are the answers to questions (1) through (4) above if we want to identify not just one stiffness k_i but a group of them, say k_1 and k_5 ?

Figure 10(A) shows how the information on the stiffeness parameter k_1 changes with time for records obtained at various locations when an impulsive force is applied at mass m_1 with I= 10 units. As seen from the figure, the information obtained from Location 2 (see Figure 9(A)) is the maximum and this says that the sensor (for identification of k_1), if only one such sensor be available, should be placed at Location 2. We note that our intuitive idea of using a sensor at Location 1 would have provided about 36% less information than the optimal sensor location

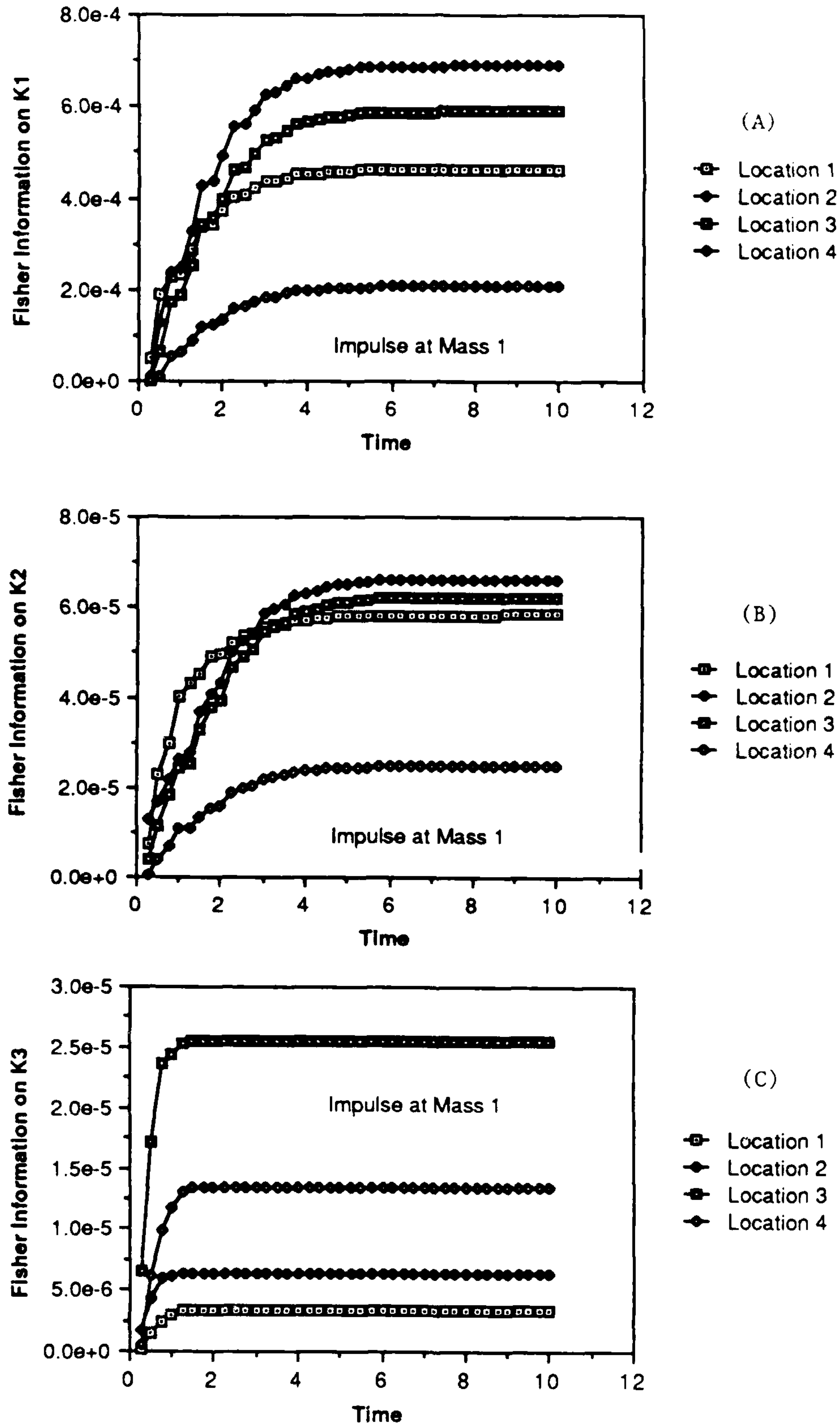
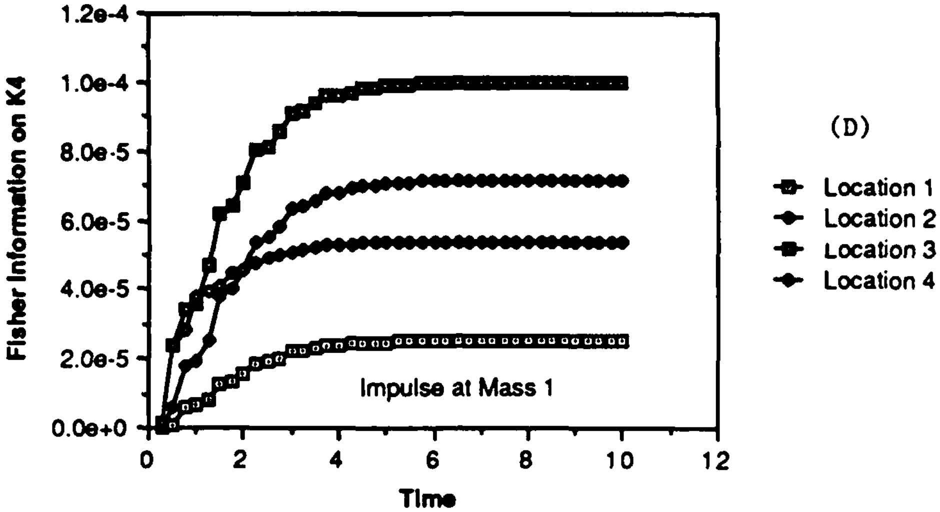


Figure 10. Optimal Sensor Locations for Impulsive Force Applied to Location No. 1



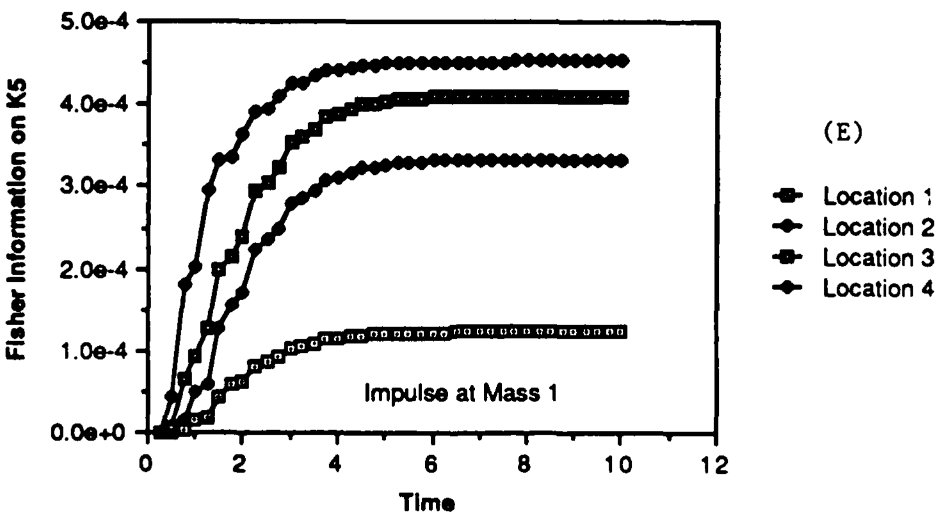


Figure 10. Optimal Sensor Locations for Impulsive Force Applied to Location No. 1

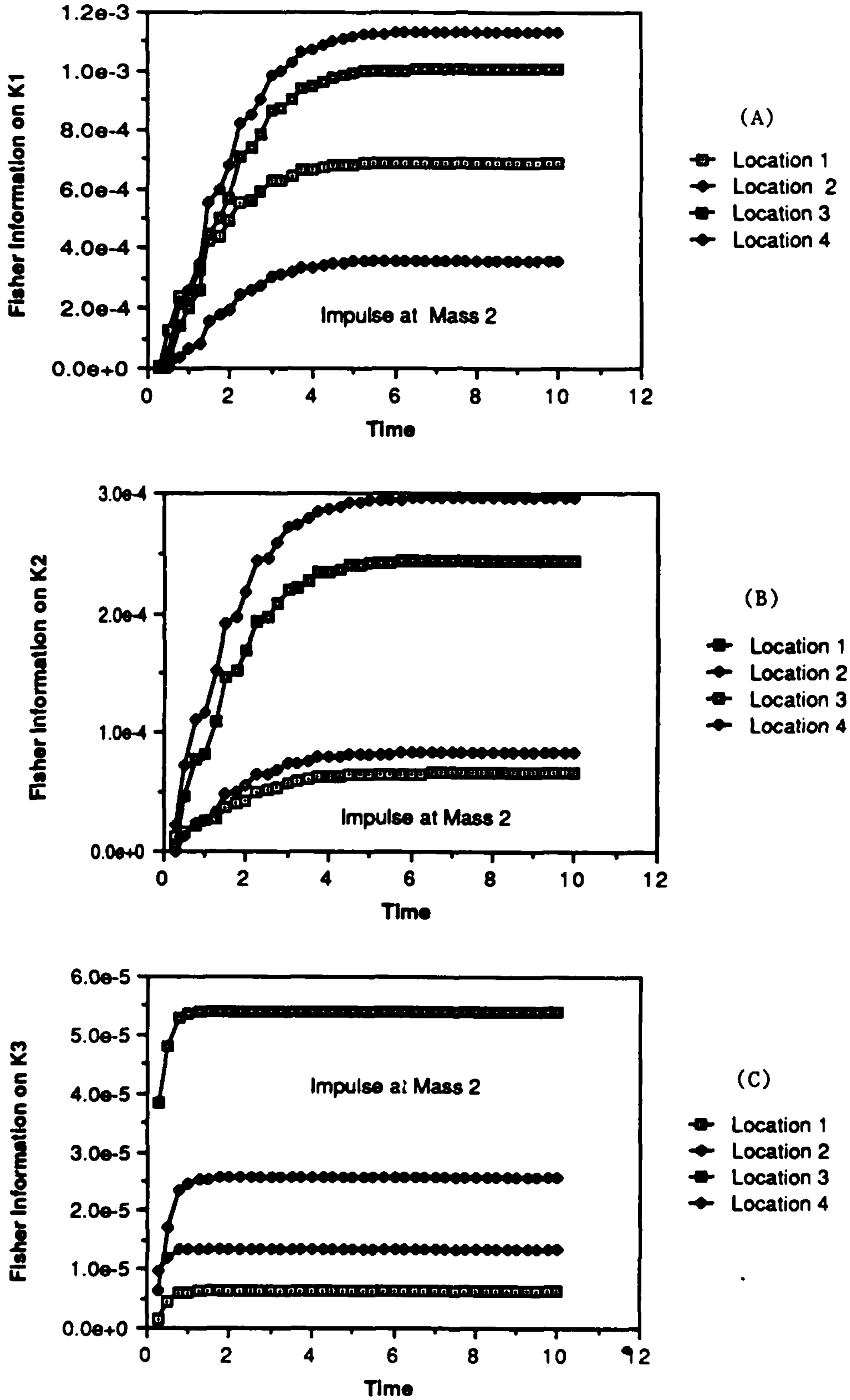


Figure 11. Optimal Sensor Locations for Impulsive Force Applied to Location No. 2

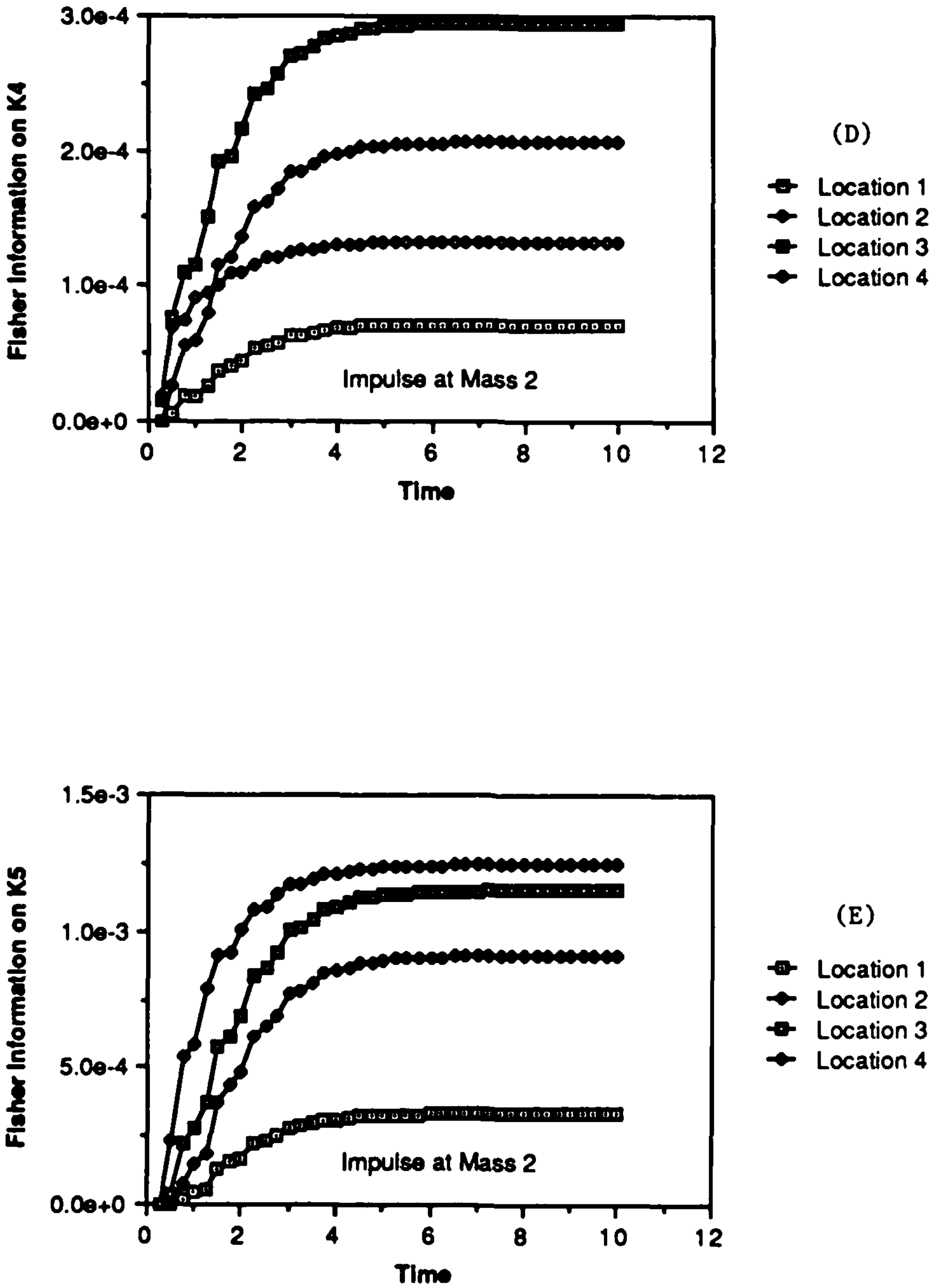


Figure 11. Optimal Sensor Locations for Impulsive Force Applied to Location No. 2

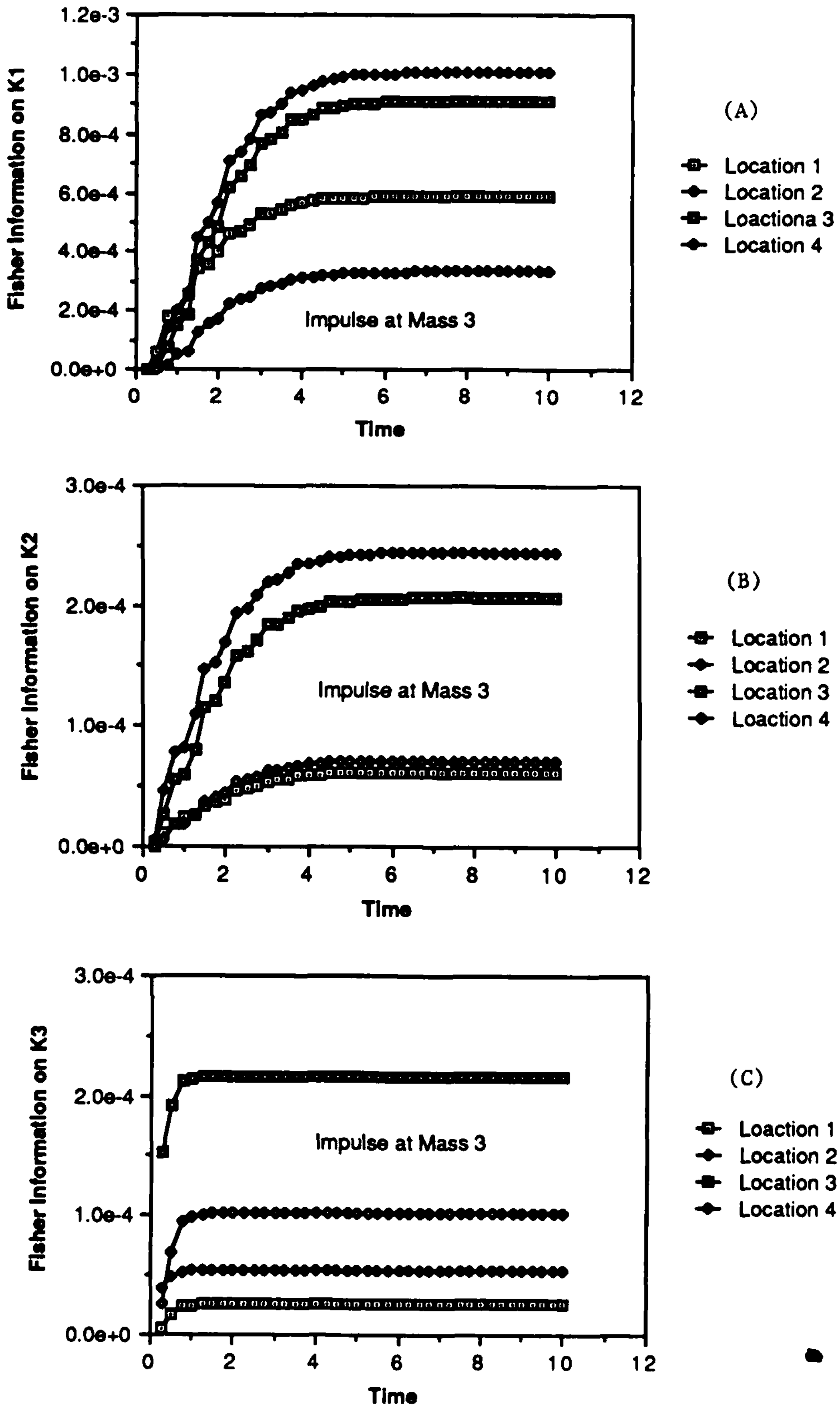
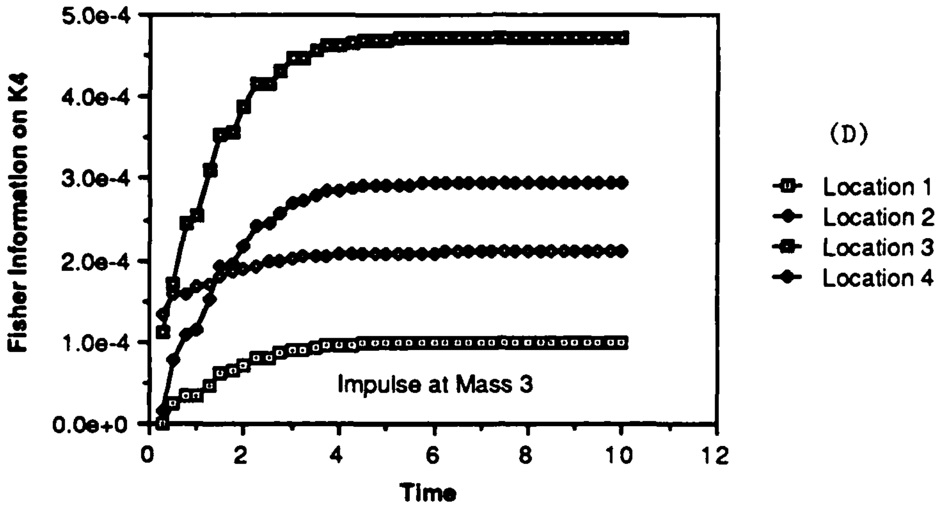


Figure 12. Optimal Sensor Locations for Impulsive Force Applied to Location No. 3



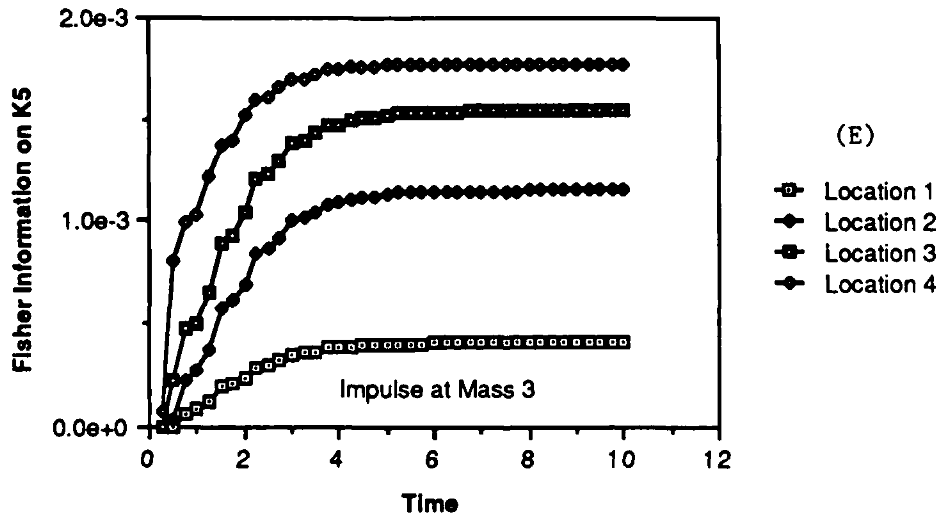


Figure 12. Optimal Sensor Locations for Impulsive Force Applied to Location No. 3

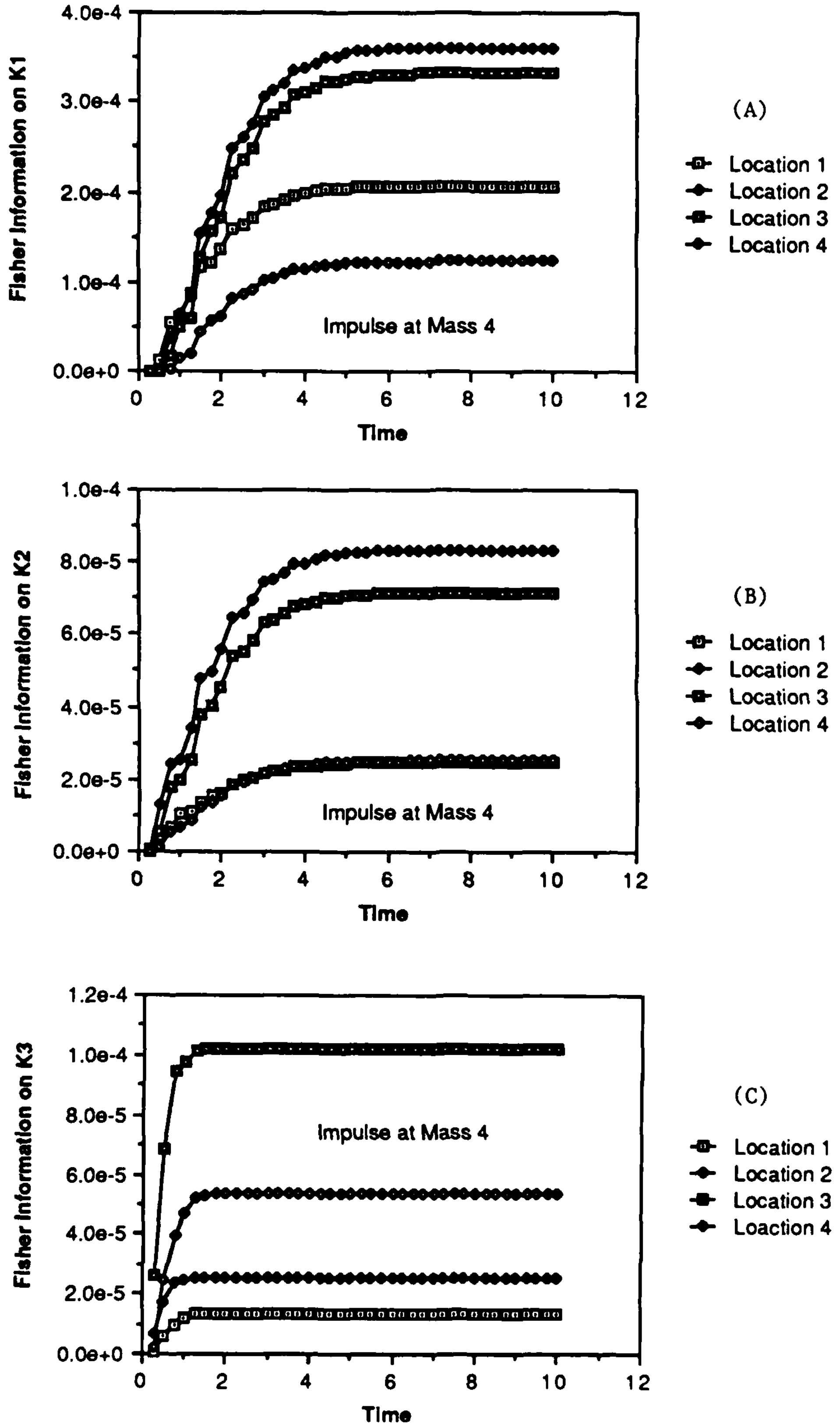
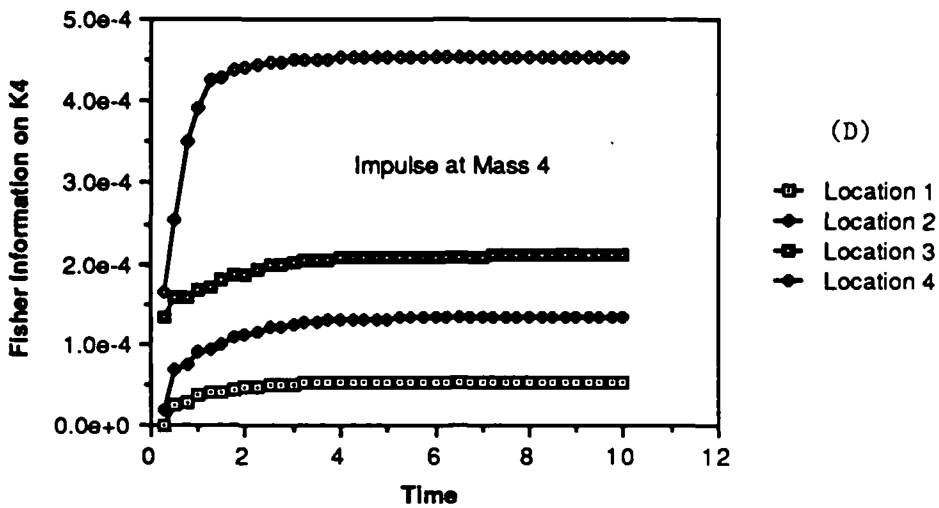


Figure 13. Optimal Sensor Locations for Impulsive Force Applied to Location No. 4



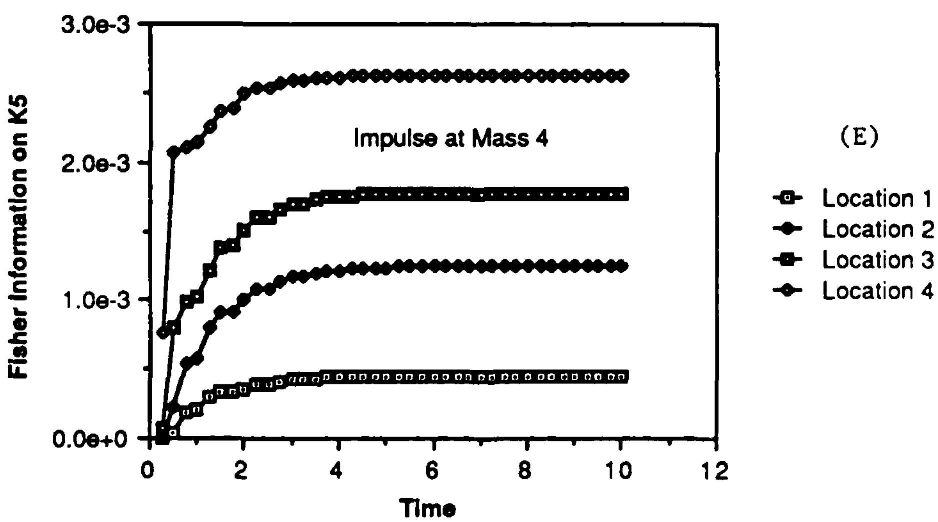


Figure 13. Optimal Sensor Locations for Impulsive Force Applied to Location No. 4

obtained. The graph also gives the rank ordering for optimal sensor locations as { Location 2, Location 3, Location 1, Location 4}, Location 2 being the best, Location 4 being the worst.

Figures 10(B), 10(C),10(D) and 10(E) indicate the optimal sensor locations(OSL) for identification of the parameters k_2, k_3, k_4, k_5 respectively using an impulsive force applied at Location 1. We note that the OSL depend on the parameter that is required to be identified. Also of interest is the fact that the Fisher Information, Q(t), at each location is a function of time. Thus Figure 10(B) shows that were k_2 to be identified using simply a two second length of record beginning at zero time, then Location 1 would be the optimal location. However, the use of a longer duration of record for identification of k_2 would yield Location 2 as the optimal location as seen in Figure 10(B).

Figures 11, 12 and 13 show similar results for identification of the stiffness parameters using an impulsive force (with I = 10) applied at masses m_2, m_3 and m_4 respectively. We see that the extent of information obtained about a parameter for the purposes of identifying it and therefore, in general, the optimal sensor locations, depend on the location where the force is applied. Thus figures 10(A) and 11(A) show that the information about the parameter k_1 from measurements taken at Location 2 is about 1.7 times greater if the impulse is applied at mass m_2 rather than at mass m_1. In fact figures 10(A),11(A), 12(A) and (13a) show that to identify k_1 using one sensor, the best location for both applying the impulsive force and for obtaining a measurement record, is Location 2.

Figures 14(A) indicates results for the situation where both k_1 and k_5 are to be simultaneously

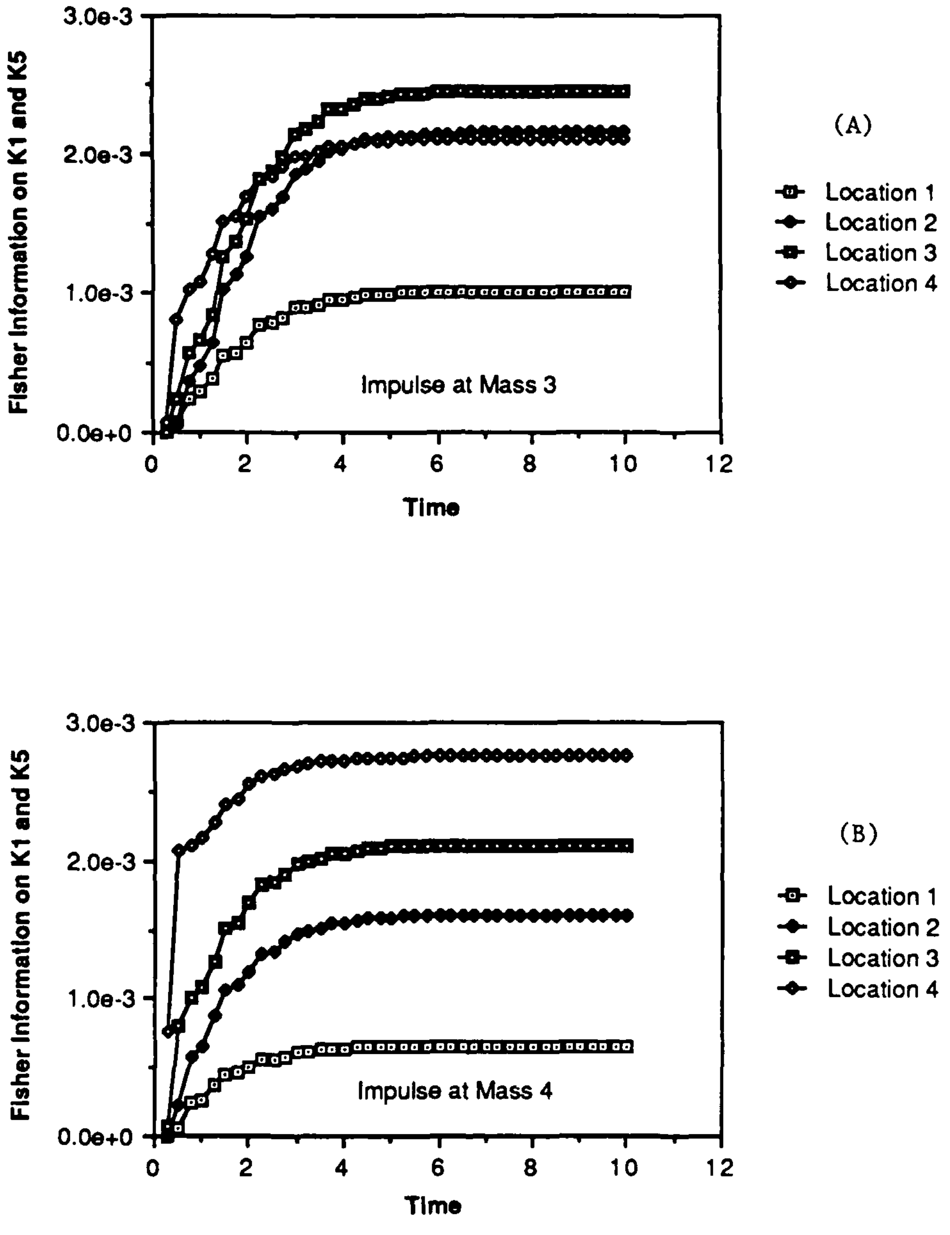
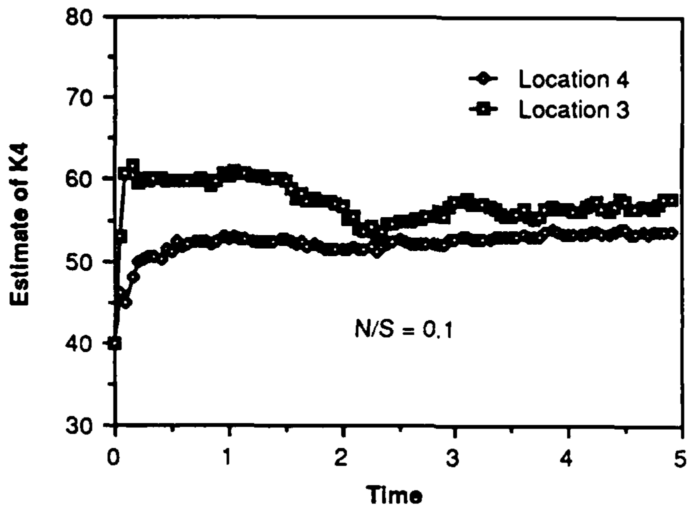


Figure 14. Optimal Sensor Locations for Simultaneous Identification of k_1 and k_5

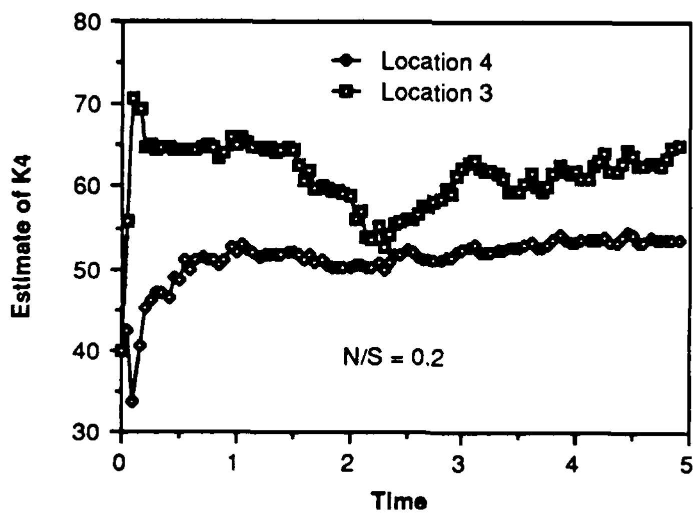
identified using noisy measurements at one or more locations with an impulsive force (I=10) applied at mass m_3. While the identification of k_1 alone would show the OSL to be at Location 2 (Figure 12(A)), and that of k_5 alone to be Location 5 (Figure 12(E)), the OSL for simultaneous identification of both these parameters is Location 3. The Locations can be rank ordered as { Location 3, Location 2, Location 4, and Location 1}, Location 3 being the best. Should more sensors be available, they would then successively populate the mass Locations as per this ordering so that identification of these two parameters can be best carried out. Figure 14(B) shows a similar result except that the impulse (I =10) is applied now at mass m_4. We observe that the rank ordering of locations, as per our Trace criterion, has now significantly changed to {Location 4, Location 3, Location 2, Location 1}, Location 4 being the best. Having answered the five questions that were posed above, we next go on to verify some of our results.

3.9.2. Verification of Optimal Sensor Locations as Determined by the Methodology

Consider the results depicted in Figure 13(D) where the OSL is obtained for the situation where an impulsive force (I=10) is applied to mass m_4 and identification of k_4 is intended. The system parameters are those chosen in Section 3.9.1. The correct value for k_4 is 50 units. The results show (Figure 13(D)) that Location 4 is far superior to Location 3. Parameter identification using the reccursive prediction error method (RPEM) was carried out using records obtained from Locations 3 and 4 in response to the impulsive force at m_4. For comparison purposes the same identification scheme (and computer program) was used for records obtained from both locations. The identification scheme was started off with a close-by initial guess, namely k_4=40. The results

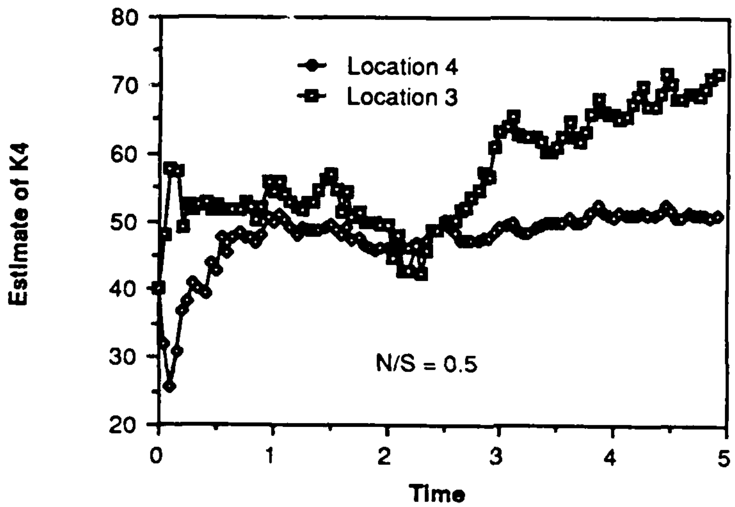


(A)

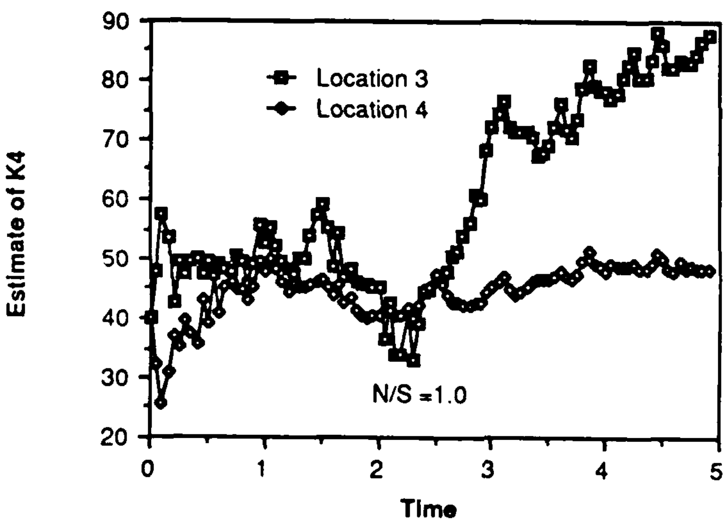


(B)

Figure 15. Verification of Optimal Sensor Location Methodology

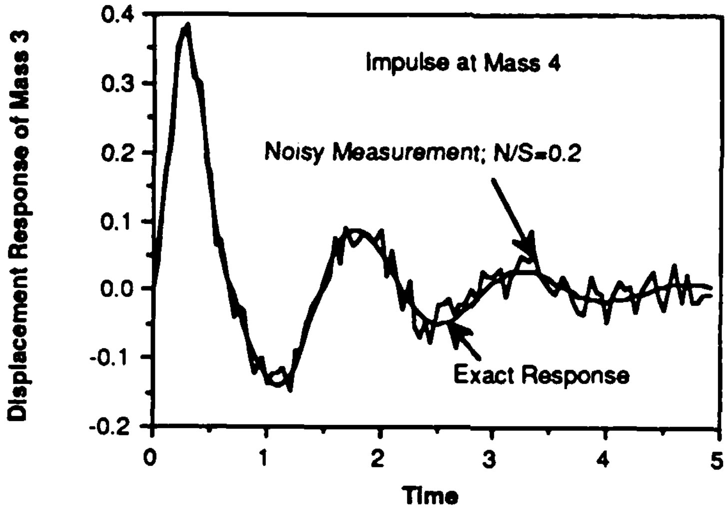


(C)

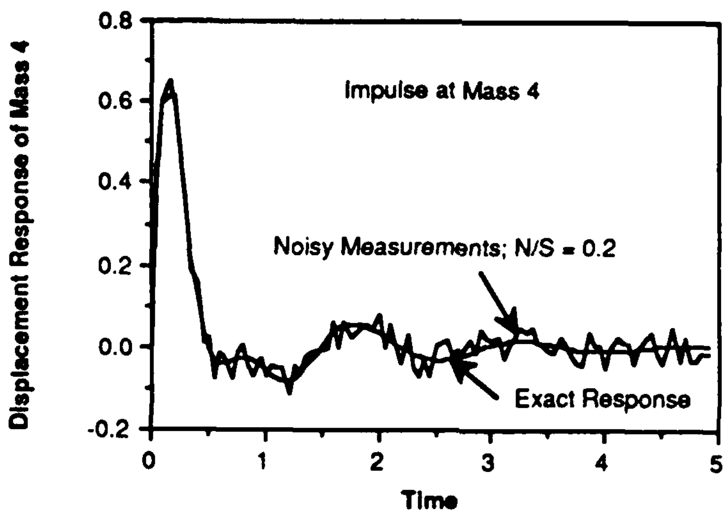


(D)

Figure 15. Verification of Optimal Sensor Location Methodology

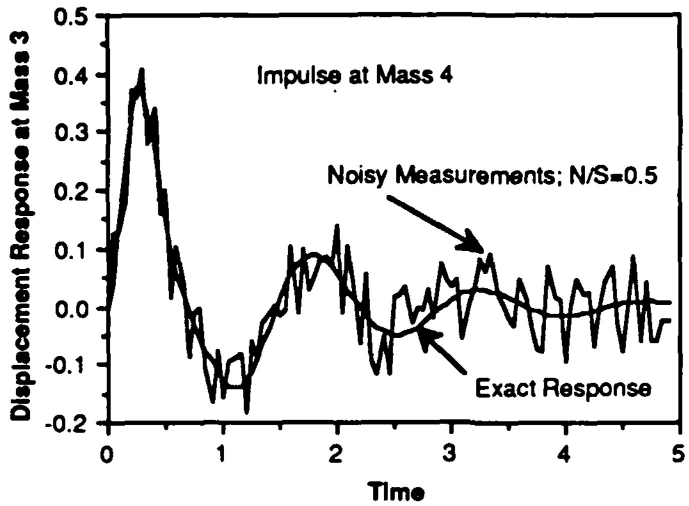


(A)

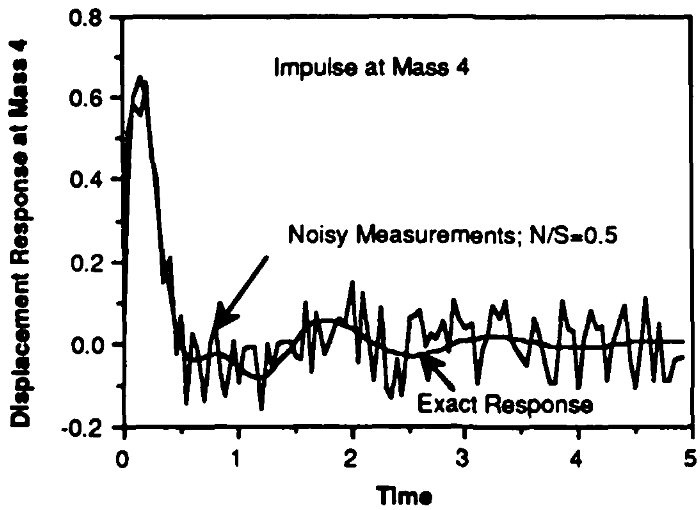


(B)

Figure 16. Noisy Response Measurements for Identification N/S=0.2

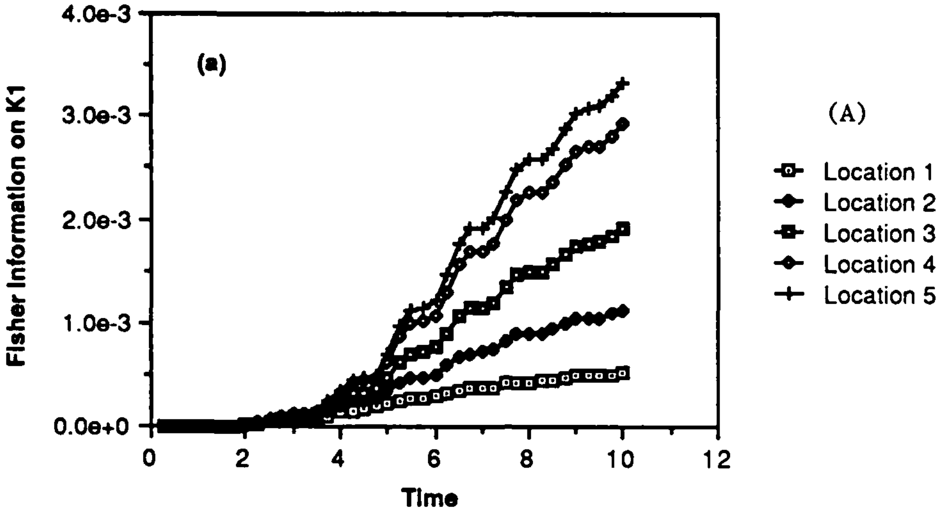


(A)

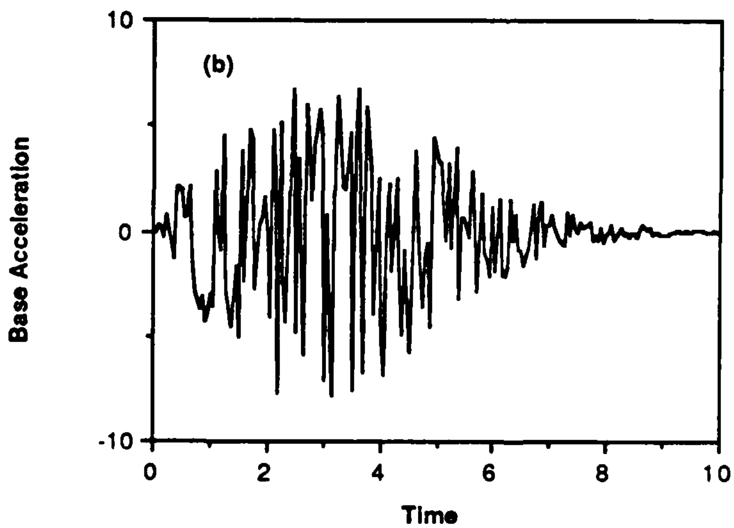


(B)

Figure 17. <u>Noisy Response Measurements for Identification N/S=0.5</u>



(A)



(B)

Figure 18. Optimum Sensor Locations for Transient Base Excitation

of this identification are shown in figures 15(A) through 15(D) for four different levels of noise-to-signal ratios (indicated in the figure by N/S). To provide a feel for the extent of noise prevalent in the records we show in Figure 16 and Figure 17 the noisy displacement records used for identification for two noise levels: N/S= 0.2 and N/S = 0.5. We note that while Locations 3 and 4 provide about the same accuracy of identification for N/S values less than 0.1, as the N/S ratio increases, Location 4 as predicted by Figure 13(D) is indeed superior. In fact identification of k_4 can be carried out with reasonable accuracy even when N/S=1.0 as seen in Figure 15(D) with measurements from Location 4. On the other hand measurements from Location 3 cause the same identification scheme to diverge for N/S=0.5 and N/S=1.0.

3.9.3. Fixed-Free System of Figure 9(B)

We present now an example of OSL for a transient excitation provided at the base of the system depicted in Figure 9(B). The system parameters are: $m_1=m_2=2$; $m_3=m_4=1$; $m_5=0.5$; $k_1=k_2=100$; $k_3=75$; $k_4=50$; $k_5=50$. The damping is again taken to be of Raleigh form (i.e. $C= 2\alpha M + 2\beta K$) with $\alpha=0.001$ and $\beta= 0.04$. Figure 18 shows the base acceleration and the results for optimal sensor location for identification of parameter k_1. It is interesting to note that though considerations of uniqueness in identification would dictate Location 1 to be optimal, considerations of identifying the parameter k_1 starting from a close-by estimate shows that Location 4 is optimal. This example therefore brings out the difference between "global" convergence and "local" convergence as discussed in Section 3.2.

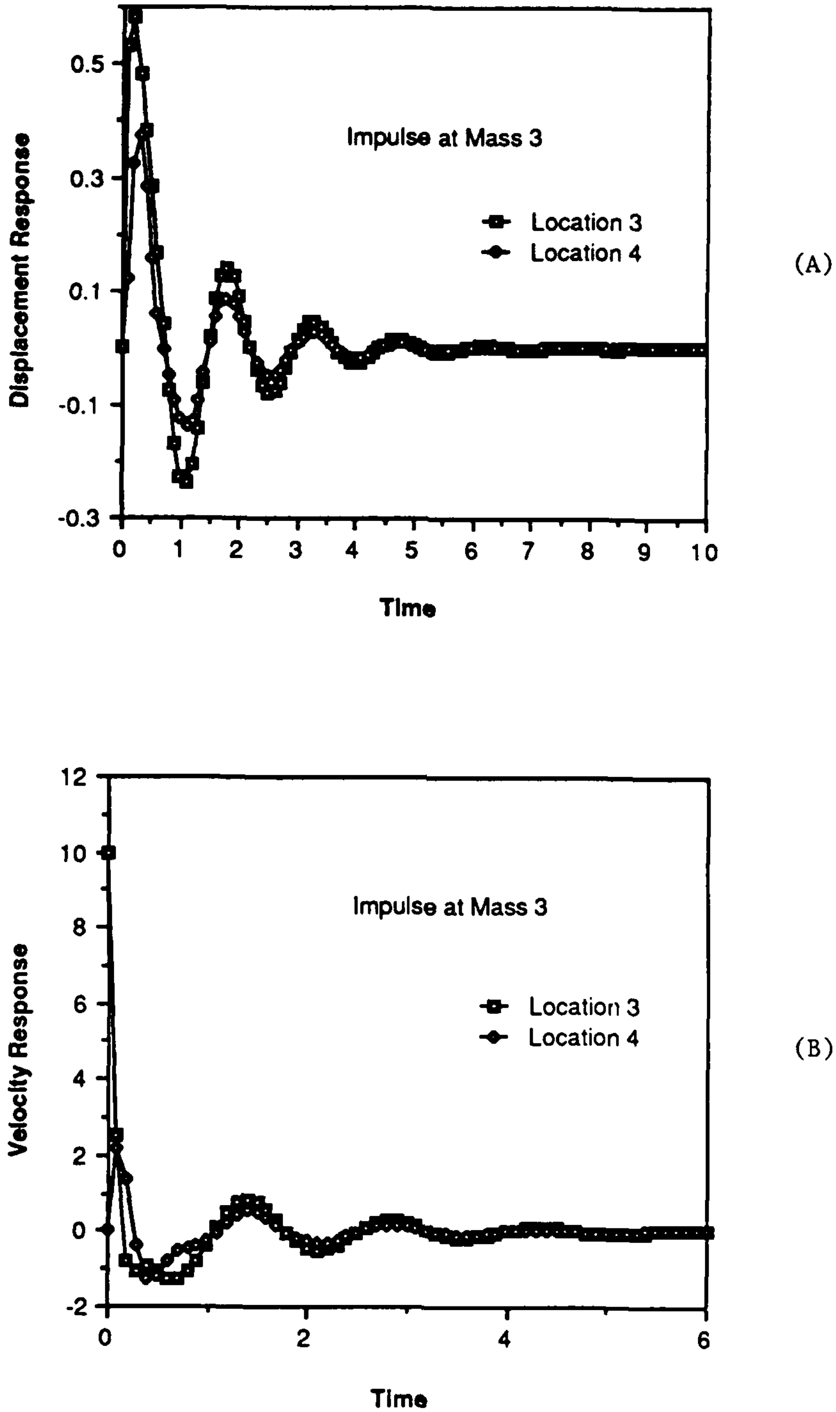
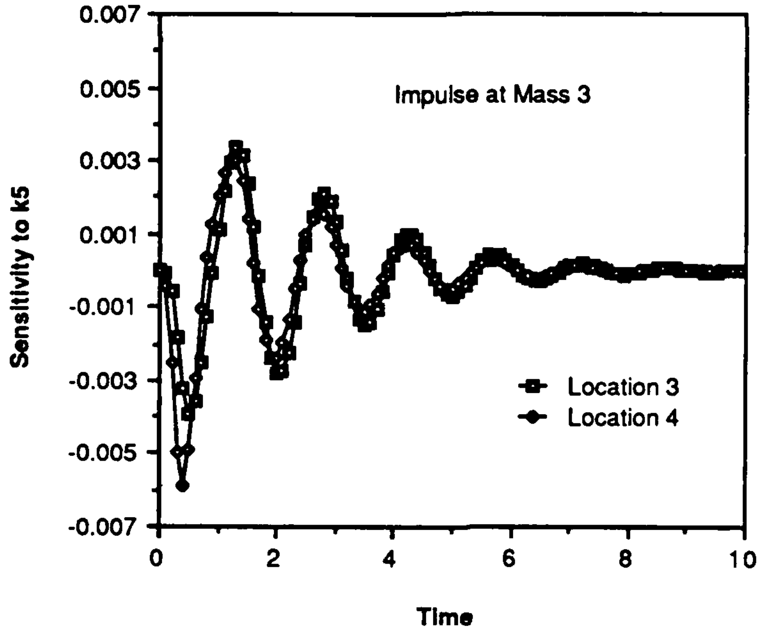
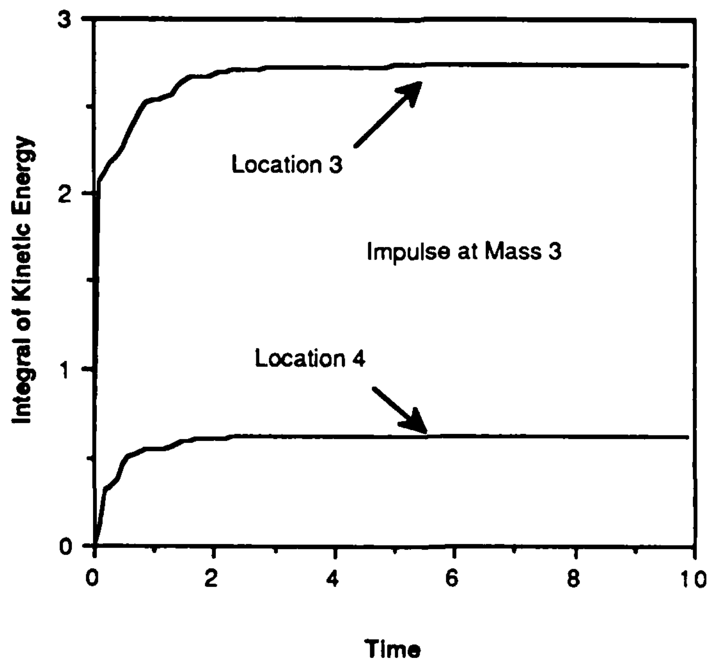


Figure 19. Response of System with Impulsive Force at Location 3



(A)



(B)

Figure 20. Comparison of Kinetic Energy and Sensitivity of Measurements

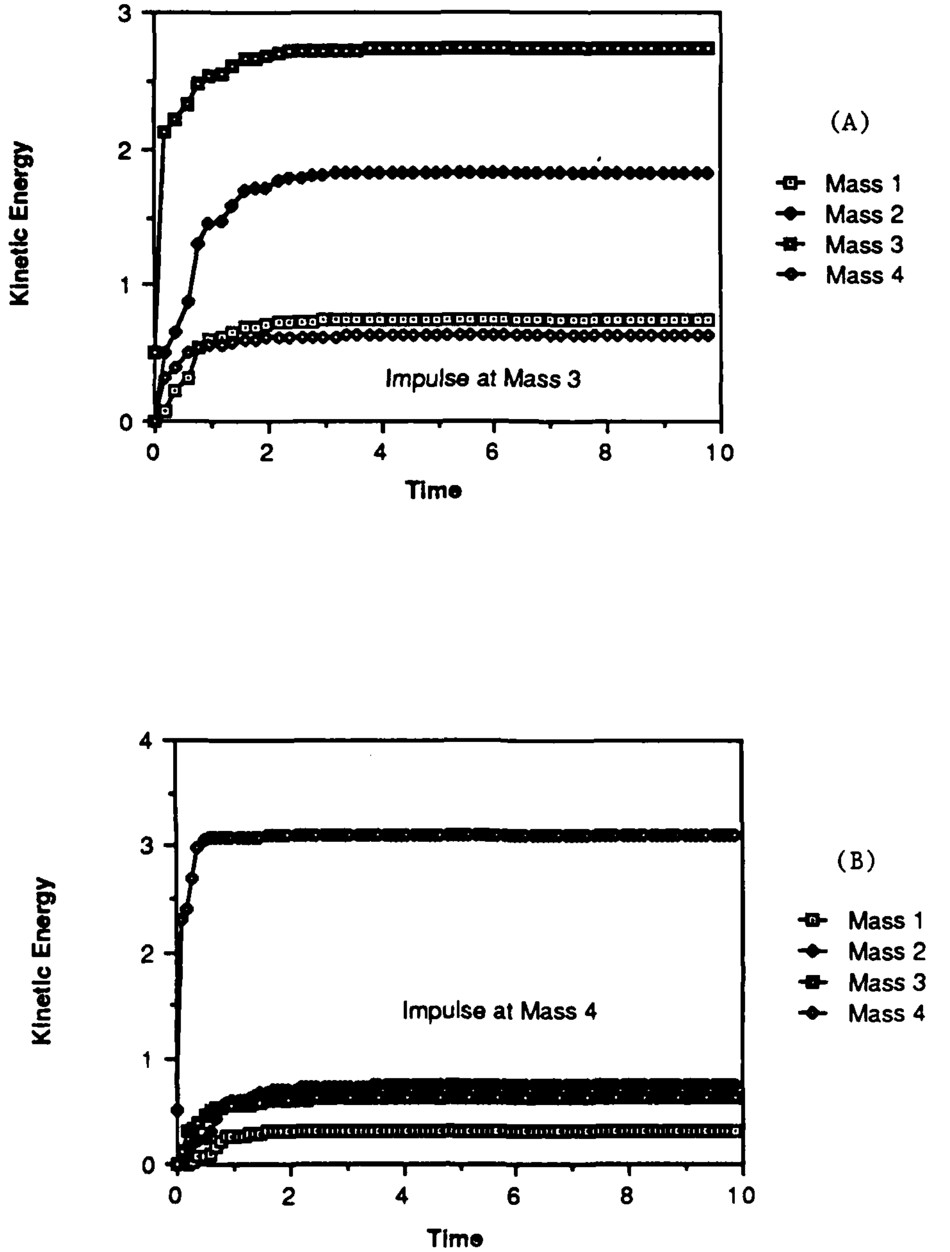


Figure 21. Results for Kinetic Energy at Various Locations

3.10. The Kinetic Energy Criterion

Some investigators have proposed that, heuristically speaking, displacement sensors be located at locations where the kinetic energy of the system is a maximum. While this may be a somewhat intuitive approach to the problem we have found that this Kinetic Energy Criterion does not yield, in general, the optimal sensor locations. Firstly, such a criterion is not dependent on the parameters that are required to be identified as any such criterion should. Secondly, and perhaps more importantly, our concern in locating sensors for best identification of parameters hinges around the sensitivity of measurements to the parameters to be identified and not on the kinetic energy of the system. For the sytem considered in Figure 9(A) the results shown in Figure 12(E) indicate that the optimal sensor location is at Location 4. Figure 19 shows the response of the system at Locations 3 and 4 to an impulsive force (I=10) applied at mass m_3. Figure 20(B) shows that the Kinetic energy(KE) at Location 4 is lower than that at Location 3. Yet Location 4 is obtained as the OSL from Figure 12(E). This is explained by figure 20(A) which shows that though the KE at Location 4 is lower than at Loaction 3, the sensitivity (actually, its absolute value) of measurement to parameter k_5 is higher at Location 4 than at Location 3. The results of the KE criterion for the cases depicted in figures 14(A) and 14(B) are shown in figures 21(A) and 21(B) respectively. Again we see that the KE criterion would lead to a different and erroneous rank ordering of locations for the simultaneous identification of k_1 and k_5.

3.11. Conclusions

We have in this section developed a methodology for optimally locating sensors for parameter identification using noisy measurement data. The methodology is predicated on starting

any such identification process with a near-by, close initial guess estimate of the parameters to be identified.

The optimal sensor locations(OSL) are shown to depend on: 1) the nature of the system (the structure of the differential equations), 2) the specific parameter values of the different parameters in the system model, 3) the number of sensors to be used, 4) the duration of time over which the identification is to be carried out, 5) the specific parameters to be identified, and 6) the nature and location(s) of the input time functions (applied forces). A simple algorithm, which is computationally efficient, has been developed for obtaining the OSL.

The methodology has been applied to a specific set of multi-degree-of-freedom systems and numerical results have been obtained. The methodology provides a rank ordering of the locations from best to worst. It also answers, in a rational manner, where to locate additional instruments, given that several are already in place in a dynamic system.

The methodology has been validated (though in a limited sense only) by actually using data from various locations and showing that those locations that are predicted to be optimal by the methodology do indeed provide tha best identification of the parameters concerned from noisy measurement data.

The results have shown that the heuristically obtained Kinetic Energy Criterion, which is sometimes alluded to in the literature, has little to do with optimally locating sensors and, to that extent, is inappropriate for use in developing methodologies relevant to such problems.

SECTION 4. PRELIMINARY RESULTS IN DATA COMPRESSION

This section deals with finding the optimal measurement locations for a structural system modelled by a single-degree-of-freedom oscillator, so that any one of the parameters to be identified can be estimated with a minimum variance. The measurements are assumed to be taken in a noisy environmment, and the section addresses both linear and nonlinear, nonhysteretic systems. Besides the analytical relations deduced for the optimal measurement locations, it is found that, in general, there may exist measurement locations at which no additional information on the parameter under consideration is generated. For the linear case, the optimal measurement locations are found to be independent of the system response and the actual values of the parameters to be identified. They solely depend on the nature of the excitation used in the identification prodecure. Analytical results relating to the optimal measurement locations for minimizing the sum of the variances of the estimates of some of the parameters are also provided.

4.1. Introduction

The identification of parameters in dynamic models for structural systems is a field that is rapidly gaining importance. In this section, we attempt to study the optimal spacing of measurements for a structural system modelled by a single-degree-of-freedom oscillator so that the variance of one of the parameters being identified is minimized.

We start with a linear oscillator and, using Fourier transforms, derive a set a linear

algebraic equations. The condition on the measurement frequencies so that the estimated variance (from noisy measurement data) of either the mass parameter, the stiffness parameter or the damping parameter is minimal is derived analytically. It is also found that there may exist a set of frequencies ω, at which no additional information on that parameter is available, yielding no reduction in its estimated variance.

The determination of the optimal measurement frequencies depends solely on the nature of the forcing function used in the identification procedure and is invariant with respect to the actual values of the parameters being estimated.

A numerical example is indicated to illustrate the analytically obtained results. The method is then extended to find the optimal measurement times for structural systems modelled by general nonlinear, second-order differential equations which represent memoryless systems. It is shown, again, that a set of time points may exist at which measurements, if made, will not yield additional information on the parameter of specific concern.

Though the results obtained at this time are pruely analytical, it is anticipated that they will help in the design of experiments, especially where data handling and reduction are a major cost concern.

4.2. Problem statement

Consider a structure modelled by a single-degree-of-freedom system oscillator

subjected to an excitation force $q(t)$. If x is the displacement of such an oscillator, then its equation of motion is

$$m\ddot{x} + c\dot{x} + kx = q(t), \quad x(0) = 0,\ \dot{x}(0) = 0, \tag{152}$$

where the parameters m, c and k denote the mass, damping and stiffness respectively and are assumed to be real numbers. Taking fourier transforms, this yields

$$-m\omega^2 X(\omega) + ic\omega X(\omega) + kX(\omega) = Q(\omega), \tag{153}$$

so that

$$1/X(\omega) = (-m\omega^2 + ic\omega + k)/Q(\omega), \qquad \omega \varepsilon I_\Omega, \tag{154}$$

where we shall assume that the division on both sides of equation (153) is possible i.e., there exists an open interval, I_Ω, such that for all $\omega \in I_\Omega$, $X(\omega)$ and $Q(\omega)$ are not identically zero.

We shall assume that the parameters m, k and c need to be identified, and we shall direct our interest to finding if there exists a set of frequencies $\omega_{i,}$ $i = 1, 2, \ldots$ such that the variance of any one of the desired parameters (i.e., m, or k or c) can be minimized by using the data i.e., $X(\omega)$ and $Q(\omega)$ at those specific frequencies.

Relation (154) can be rewritten, after separating the real and imaginary parts, as

$$\begin{bmatrix} U(\omega) \\ V(\omega) \end{bmatrix} = \begin{bmatrix} -\omega^2 f(\omega) & -\omega g(\omega) & f(\omega) \\ -\omega^2 g(\omega) & \omega f(\omega) & g(\omega) \end{bmatrix} \begin{bmatrix} m \\ c \\ k \end{bmatrix} + \begin{bmatrix} \varepsilon_U(\omega) \\ \varphi_V(\omega) \end{bmatrix}, \tag{155}$$

where we have represented

$$1/X(\omega) = U(\omega) + iV(\omega),$$
$$1/Q(\omega) = f(\omega) + ig(\omega), \tag{156}$$

and the measurements U (ω) and V (ω) are corrupted by measurement noise ε_U (ω) and $\varepsilon_V(\omega)$. Defining

$$[a_j(\omega)] \overset{\Delta}{=} a(\omega) \overset{\Delta}{=} [-\omega^2 f(\omega)\ -\omega g(\omega)\ f(\omega)]^T \tag{157a}$$

and

$$[b_j(\omega)] \overset{\Delta}{=} b(\omega) \overset{\Delta}{=} [-\omega^2 g(\omega)\ \omega f(\omega)\ g(\omega)]^T \tag{157b}$$

for each value of $\omega = \omega_i$, i = 1, 2,..., N, equation (155) can be written and the BLUE estimator obtained. This can be expressed by the relation

$$z = H\theta + \varepsilon, \tag{158}$$

where

$$z = [U(\omega_1)V(\omega_1)U(\omega_2)V(\omega_2)\cdots U(\omega_n)V(\omega_n)]^T,$$
$$H = [a(\omega_1)\ b(\omega_1)\ a(\omega_2)\ b(\omega_2)\cdots a(\omega_n)\ b(\omega_n)]^T,$$
$$\varepsilon = [\varepsilon_U(\omega_1)\ \varepsilon_Y(\omega_2)\ \varepsilon_V(\omega_2)\cdots \varepsilon_U(\omega_n)\ \varepsilon_V(\omega_n)]^T. \tag{159}$$

Assume that the error vector ε has the statistic

$$E[\varepsilon] = 0 \text{ and } R = E(\varepsilon\varepsilon^T) = \mathrm{diag}(\sigma_1{}^2, \sigma_1{}^2, \sigma_2{}^2, \ldots, \sigma_n{}^2, \sigma_n{}^2), \tag{160}$$

where $\sigma_i = \sigma(\omega_i)$, $i = 1, 2,\ldots, n$. The covariance of the BLUE estimate of the vector θ becomes [16]

$$P = (H^T R^{-1} H)^{-1}. \tag{161}$$

where P is a 3×3 matrix whose diagonal elements P_{11}, P_{22} and P_{33} are the variances in the estimate of m, c and k respectively.

4.3. Optimal choice of frequencies, ω_k.

We shall now attempt to choose the frequencies $\omega_k \in I_\Omega$ in such a way that the i th element of P, P_{ii}, is minimized. To that end we first differentiate relation (161) with respect to ω_k to yield

$$\frac{\partial P}{\partial \omega_k} = -P\left[\frac{\partial H^T}{\partial \omega_k} R^{-1} H + H^T R^{-1} \frac{\partial H}{\partial \omega_k} - H^T R^{-1} \frac{\partial R}{\partial \omega_k} R^{-1} H\right] P, \tag{162}$$

so that

$$\left(\frac{\partial P}{\partial \omega_k}\right)_{\pi} = -2\left[P\frac{\partial H^T}{\partial \omega_k}R^{-1}HP\right]_{\pi} + \left[PH^TR^{-1}\frac{\partial R}{\partial \omega_k}R^{-1}HP\right]_{\pi} \tag{163}$$

Assuming that $a(\omega)$, $b(\omega)$ and $\sigma(\omega)$ are continuous functions in I_{Ω},

$$(\partial H^T/\partial \omega_k) = [0...0\ a(\omega_k)\ b(\omega_k)\ 0...0],$$

so that

$$\frac{\partial P_{\pi}}{\partial \omega_k} = -\frac{2}{\sigma_k^2}\left\{P[\dot{a}(\omega_k)a^T(\omega_k) + \dot{b}(\omega_k)b^T(\omega_k)]P\right\}_{\pi}$$

$$+\frac{2\dot{\sigma}(\omega_k)}{\sigma^3(\omega_k)}\left\{P[a(\omega_k)a^T(\omega_k) + b(\omega_k)b^T(\omega_k)]\,P\right\}_{\pi} \tag{164a}$$

On expanding, equation (164a) becomes

$$\frac{\sigma_k^2}{2}\frac{\partial P_{\pi}}{\partial \omega_k} = -\sum_{j,s}\left\{P_{rj}\dot{a}_j(\omega_k)a_s(\omega_k)P_{sr} + P_{rj}\dot{b}_j(\omega_k)b_s(\omega_k)P_{sr}\right\}$$

$$+\frac{\dot{\sigma}(\omega_k)}{\sigma(\omega_k)}\left\{\left(\sum_j P_{rj}a_j\right)^2+\left(\sum_j P_{rj}b_j\right)^2\right\}. \tag{164b}$$

The condition that P_{rr} be extremal then yields

$$\left[\sum_j P_{rj}\dot{a}_j(\omega_k)\right]\left[\sum_s P_{sr}a_s(\omega_k)\right]+\left[\sum_j P_{rj}\dot{b}_j(\omega_k)\right]\left[\sum_s P_{sr}b_s(\omega_k)\right]$$

$$-\frac{\dot{\sigma}(\omega_k)}{\sigma(\omega_k)}\left[\left(\sum_j P_{rj}a_j\right)^2+\left(\sum_j P_{rj}b_j\right)^2\right]=0. \tag{165}$$

LEMMA 1. If a real $\omega_k \in I_\Omega$ exists such that $f(\omega_k)$ and $g(\omega_k)$ are not zero, and for any $r \in [1,2,3]$,

$$\sum_s P_{sr}a_s(\omega_k) = 0, \tag{166}$$

then,

$$\sum_s P_{sr}b_s(\omega_k) = 0, \text{ for that value of r,} \tag{167}$$

and vice versa.

Proof. Let us say that we have a sequence of ω's, $\omega_1, \omega_2, \ldots, \omega_k, \ldots, \omega_N$, where for ω_k relation (166) is valid. Using relations (159) and (161) we have,

$$P^{-1} = \sum_{s=1}^{N} \left[\frac{a(\omega_s)a^T(\omega_s) + b(\omega_s)b^T(\omega_s)}{\sigma_s^2} \right],$$

so that

$$P^{-1} = \begin{bmatrix} \sum_{s=1}^{N} \frac{\omega_s^4 \alpha^2(\omega_s)}{\sigma_s^2} & 0 & -\sum_{s=1}^{N} \frac{\omega_s^2 \alpha^2(\omega_s)}{\sigma_s^2} \\ 0 & \sum_{s=1}^{N} \frac{\omega_s^2 \alpha^2(\omega_s)}{\sigma_s^2} & 0 \\ -\sum_{s=1}^{N} \frac{\omega_s^2 \alpha^2(\omega_s)}{\sigma_s^2} & 0 & \sum_{s=1}^{N} \frac{\alpha^2(\omega_s)}{\sigma_s^2} \end{bmatrix}, \qquad (168)$$

where

$$\alpha^2(\omega_s) = f^2(\omega_s) + g^2(\omega_s). \qquad (169)$$

The determinant Δ of P^{-1} then becomes

$$\Delta = \sum \omega_s^2 \alpha^2(\omega_s) \cdot \left\{ \sum_{s=1}^{N} \frac{\alpha^2(\omega_s)}{\sigma_s^2} \sum_{s=1}^{N} \frac{\omega_s^4 \alpha^2(\omega_s)}{\sigma_s^2} \left[\sum \frac{\omega_s^2 \alpha^2(\omega_s)}{\sigma_s^2} \right]^2 \right\}, \qquad (170)$$

which by the Cauchy-Schwartz inequality is always > 0, as it should be, since P is a covariance matrix. Thus the matrix P now becomes

$$P = \begin{bmatrix} AB & 0 & B^2 \\ 0 & CA - B^2 & 0 \\ B^2 & 0 & BC \end{bmatrix} / \Delta, \qquad (171a)$$

where

$$A = \sum_{s=1}^{N} \frac{\alpha^2(\omega_s)}{\sigma_s^2}, \quad B = \sum_{s=1}^{N} \frac{\omega_s^2 \alpha^2(\omega_s)}{\sigma_s^2} \qquad (171b)$$

$$C = \sum_{s=1}^{N} \frac{\omega_s^4 \alpha^2(\omega_s)}{\sigma_s^2}, \qquad (171c)$$

$$\Delta = B(AC - B^2). \qquad (172)$$

Noting relation (157), condition (166) becomes

$$\begin{aligned} f(\omega_k)[-\omega_k^2 A + B] &= 0, \quad r=1 \\ -g(\omega_k)\,[\omega_k(CA - B^2)] &= 0, \quad r = 2, \\ f(\omega_k)[-\omega_k^2\ B + C] &= 0, \quad r = 3, \end{aligned} \qquad (173a)$$

Relation (167) yields

$$g(\omega_k)\,[-\,\omega_k^2\ A + B] = 0, \qquad r = 1,$$
$$f(\omega_k)\,[\omega_k\,(CA - B^2)] = 0, \qquad r = 2, \qquad (173b)$$
$$g(\omega_k)\,[-\,\omega_k^2\ B + C] = 0, \qquad r = 3.$$

If $g(\omega_k)$ and $f(\omega_k) \neq 0$ then two sets of equations become identical, and the result follows. ■

LEMMA 2.

$$\frac{\Sigma P_{rj} b_j}{\Sigma P_{rj} a_j} = \frac{g(\omega_k)}{f(\omega_k)}, \qquad \sum P_{rj} a_j \neq 0, r = 1,3 \qquad (174)$$

and

$$\frac{\Sigma P_{rj} b_j}{\Sigma P_{rj} a_j} = \frac{f(\omega_k)}{g(\omega_k)}, \qquad \sum P_{rj} a_j \neq 0, r = 2. \qquad (175)$$

Proof. The proof follows from equations (173a) and (173b). ■

THEOREM 1. For a given forcing function $Q(\omega)$ and any $r \in \{1,2,3\}$, there may exist frequencies ω_k such that the inclusion of data at those frequencies does not yield any improvement in the variance P_{rr} of our estimate of parameter r. Specifically, when $\omega_k \in I_\Omega$ satisfies equation (166), $\omega = \omega_k$ is such a frequency.

Proof. Let us imagine that the measurements at the frequencies $\omega_1, \omega_2, \ldots, \omega_{k-1}$ have been made and that with each measurement, the covariance matrix P is updated. After making the k th measurement at $\omega = \omega_k$, the updated covariance matrix becomes [16]

$$P^+ = P^- - P^- H_k^T \,[R_k + H_k P^- H_k^T \,]^{-1} H_k P^-, \tag{176}$$

where,

P^- denotes the covariance before the measurement at ω_k,
$P+$ denotes the covariance after the measurement at ω_k,
H_k denotes $[a,b]^T$ evaluated at ω_k, and
$R_k = \sigma_k^2$ diag (1,1).

Relation (176) can be rewritten using the notation

$$L \stackrel{\Delta}{=} P^- H_k^T \tag{177}$$

as

$$P^+ = P^- - L\,[R_k + H_k P^- H_k^T]^{-1} L^T. \tag{178}$$

Using relations (157) and (159), we have

$$L = \begin{bmatrix} \Sigma P^-_{1j} a_j & \Sigma P^-_{1j} b_j \\ \Sigma P^-_{2j} a_j & \Sigma P^-_{2j} b_j \\ \Sigma P^-_{3j} a_j & \Sigma P^-_{3j} b_j \end{bmatrix} \tag{179}$$

If relation (166) is valid for some r, then

$$\sum P^-_{rj}\, a_j\,(\omega_k) = \sum P^-_{rj}\; b_j(\omega_k) \tag{180}$$

and by relation (179)

$$L_{rj} = 0, \quad j = 1, 2 \text{ for that r.} \tag{181}$$

Consequently, from equation (178) we find

$$P^+_{rj} = P^-_{rj} \quad j = 1,2,3. \tag{182}$$

We note in passing, using relations (180) and (182),

$$\Sigma P^+_{rj} a_j = \Sigma P^+_{rj} b_j = 0. \tag{183}$$

COROLLARY 1. For values of ω_k for which $\Sigma P_{rj} a_j(\omega_k) = 0$ (for r = 1,3), the variance of the estimate of the r th variable as well as its covariance are unaffected by the measurement $\omega = \omega_k$.

Proof. The result follows directly from relation (182). ■

We note that the frequencies ω_k, which do not contain any further information (for any given r), do not depend on the parameter values m, c and k. They are only governed by the nature of the forcing function $Q(\omega)$ and can be calculated before even the measurements are made. They also do not depend on the measured responses.

COROLLARY 2a. If $g(\omega_k) \neq 0$ and $f(\omega_k) \neq 0$, there exists no $\omega_k \in I_\Omega$ except possibly $\omega_k = 0$ for which

$$\sum P_{2j} a_j(\omega_k) = 0, \tag{184}$$

provided the covariance matrix is nonsingular.

Proof. Referring to equations (173) for r = 2 and noting that $AC - B^2 \neq 0$ because the determinant of the covariance matrix is nonzero, the equations can only be satisfied by $\omega_k = 0$.

COROLLARY 2b. If $g(\omega_k) \neq 0$ and $f(\omega_k) \neq 0$, there always exists an, ω_k, $\omega_k \in (0, \infty)$, for which $\sum P_{rj} a_j(\omega_k) = 0$, r = 1, 3.

Proof. We shall show the case for r = 1. Using relation (173a), $\omega_k^2 = B/A$. From equation (172), B > 0 and A > 0. Thus $\omega_k = (B/A)^{1/2}$.
The proof for r = 3 follows along similar lines. ■

THEOREM 2. The optimal locations for the measurements $\omega = \omega_k$, $\omega_k \in I\,\Omega$, if they exist at all, which minimize the variance in the estimates P_{rr}, satisfy the following relations:

$$\Sigma P_{rj}\dot{a}_j(\omega_k)\ \frac{g(\omega_k)}{f(\omega_k)}\ \Sigma P_{sr}\dot{b}_s(\omega_k) - \frac{\dot{\sigma}(\omega_k)}{\sigma(\omega_k)}\left[1+\left(\frac{g(\omega_k)}{f(\omega_k)}\right)^2\right]\Sigma P_{rj}a_j = 0, \qquad r = 1,3 \quad (185a)$$

when $f(\omega_k) \neq 0$. When $f(\omega_k) = 0$, they satisfy the relation

$$\sum_s P_{sr}\dot{b}_s(\omega_k) - \frac{\dot{\sigma}(\omega_k)}{\sigma(\omega_k)} \sum_s P_{sr}b_s(\omega_k) = 0, \quad r = 1,3, \qquad (185b)$$

Proof. Using equation (165) and Lemma 2, the result follows. A similar result can be written for the cases when $g(\omega_k) \neq 0$ and $g(\omega_k) = 0$ respectively.

Equations (185a) and (185b) express the criteria for finding observation points ω_k such that the mass m or the stiffness k can be optimally identified. We note that the optimal location ω_k of the kth observation point depends in general upon the location of all the previous observation points as contained in P_{rj} and P_{sr}.

THEOREM 3. The optimal locations for the measurements $\omega = \omega_k$, $\omega_k \in I\,\Omega$, if they exist at all, which minimize the variance of the damping parameter c, satisfy the relation

$$\left[1 - \frac{\omega_k \sigma(\omega_k)}{\sigma(\omega_k)} \right] [g^2(\omega_k) + f^2(\omega_k)] = - \frac{\omega_\kappa}{2} \frac{d}{d\omega} [g^2(\omega) + f^2(\omega)] \Big|_{\omega_\kappa} . \tag{186}$$

Proof. Noting relation (165) and Lemma 2, the result follows. We assume that the covariance matrix is strictly positive definite.

Theorem 3 states that to optimally locate the k th measurement, relation (186) needs to be satisfied. We note that in this case the optimal location of the k th measurement does not depend on the locations of the preceeding measurements and is purely controlled by the nature of the graphs of $f(\omega)$ and $g(\omega)$.

Next let us consider the problem of minimizing the sum of the variances of ℓ out of the 3 parameters. Let s_i, $i = 1,\ldots, \ell$ be these ℓ parameters. Let Λ be the zero matrix whose nth diagonal element is unity if $n = s_i$, $i = 1,\ldots, \ell$, $s_i \in (1, 2, 3)$. We then have the followng result.

THEOREM 4. The frequencies $\omega_k \in I_\Omega$ which make

$$\sum_{i=1}^{\ell} P_{s_i s_j}, \quad \ell \leq 3, \tag{187}$$

extremal are given by the relation

$$a^T(\omega_k)P\Lambda\left[P\dot{a}(\omega_k) - \frac{\dot{\sigma}(\omega_k)}{\sigma(\omega_k)}Pa(\omega_k)\right] + b^T(\omega_k)P\Lambda\left[P\dot{b}(\omega_k) - \frac{\dot{\sigma}(\omega_k)}{\sigma(\omega_k)}\right] = 0, \qquad (188)$$

where Λ is the selection matrix as defined above.

Proof. The extremal condition is given by

$$\sum_{i=1}^{\ell} \frac{\partial P_{s_i s_j}}{\partial \omega_k} = 0. \qquad (189)$$

Using relation (164a) this becomes

$$\sum_{i=1}^{\ell} \sum_{j,s} \left[\left\{ P_{s_i j}\dot{a}_j(\omega_k)a_s(\omega_k)P_{ss_i} + P_{s_i j}\dot{b}_j(\omega_k)b_s(\omega_k)P_{ss_i} \right\} \right.$$

$$\left. - \frac{\dot{\sigma}(\omega_k)}{\sigma(\omega_k)} \left\{ P_{s_i j}a_j(\omega_k)a_s(\omega_k)P_{ss_i} + P_{s_i j}b_j(\omega_k)P_{ss_i} \right\} \right] = 0 \qquad (190)$$

from which the results follows. ■

COROLLARY 3. The extremal values of trace(P) are given by the relation

$$a^T(\omega_k)\,P^2\dot{a}(\omega_k) + b^T(\omega_k)\,P^2\dot{b}(\omega_k) = 0 \qquad . \, (191)$$

when $\dot{\sigma} = 0$. This corresponds to $R = \sigma_0 \, \mathrm{Diag}(1,1, \ldots ,1)$.

Proof. For this case $\Lambda = I$ and the result follows. ■

COROLLARY 4. If either $f(\omega_k)$ or $g(\omega_k)$ is nonzero, then there exists no ω_k, $\omega_k > 0$, for which the vectors $P\,a(\omega_k)$ and $P\,b(\omega_k)$ equal zero, provided P is nonsingular.

Proof. Noting that the vector Pa is proportional to

$$\begin{Bmatrix} f(\omega_k)[B - \omega_k^2 A] \\ g(\omega_k)[B^2 - AC] \\ f(\omega_k)[C - \omega_\kappa^2 B] \end{Bmatrix} . \tag{192}$$

where A, B and C are defined in relation (171-172), the result follows. The result for P b is along the same lines.

COROLLARY 5. If $\ell > 1$, there exists no frequency ω_k, $\omega_k \in I_\Omega$, such that for $f(\omega_k)$ and $g(w_k)$ nonzero and P nonsingular

$$\Lambda P\, a = 0. \tag{193}$$

Proof. The result follows from Corollary 4 and relations (173).

We note therefore that, in general, the data at each frequency provide information on m

and/or k.

4.4. Numerical example.

$$\text{Let } q(t) = \begin{cases} 0, & t < 0, \\ e^{-\beta t}, & t \geq 0,\ \beta > 0. \end{cases} \tag{194}$$

Then, $f(\omega) = \beta$ and $g(\omega) = \omega$. The vectors a and b become

$$a = [-\omega^2\beta \quad -\omega^2 \quad \beta]^T,$$

and (195)

$$b = [-\omega^3 \quad \omega\beta \quad \omega]^T.$$

Let us assume that measurements at $\omega = \omega_1, \omega_2, \ldots, \omega_{k-1}$ have been made and the next measurement is to be taken at ω_k. Let P^- denote the covariance matrix at the end of the first (k - 1) measurements, and $\omega_i \in [1, \infty)$. Let $\sigma(\omega_i) = \sigma_0$, $\forall i$, so that $\sigma(\omega) \equiv 0$.

If ω_k is such that $\sum P_{3j}a_j(\omega_k) = 0$, then no improvement in the variance of the stiffness estimate can be expected by obtaining the additional measurement at $\omega = \omega_k$. This condition for the forcing function (194), after some algebra, implies

$$\omega_k = [C' / B']^{1/2}, \tag{196}$$

where

$$C' = \sum_{s=1}^{k-1} \omega_s^4(\beta^2 + \omega_s^2) \text{ and } B' = \sum_{s=1}^{k-1} \omega_s^2(\beta^2 + \omega_s^2) \ . \tag{197}$$

For illustration, assume that $k = 4$ and $\beta = 1$. If the first three measurements are taken at $\omega = 1$ rad/sec., $\omega = 2$ rad/sec., $\omega = 3$ rad/ sec., and $\sigma_j = \sigma_0$, $j = 1,2,3,4$, then

$$P^- = \sigma_0^2 \begin{bmatrix} .00648 & 0 & .04274 \\ 0 & .0089 & 0 \\ .04274 & 0 & .3405 \end{bmatrix}, \tag{198}$$

$B' = 112$, $C' = 892$, and relation (196) gives $\omega_4 = 2.82$ rad/sec. Thus a measurement at $\omega = 2.82$ rad/sec. will yield no improvement of the covariance of the stiffness. In fact, from Corollary 1 we know that $P^+_{3j} = P^-_{3\ j}$, $j = 1,2,3$. Again if ω_k is such that $\sum P_{1j}a_j(\omega_k) = 0$, then the new measurement will provide no additional information on the mass parameter. After some algebra, we find that this relation gives

$$\omega_k = (B' / A')^{1/2}, \tag{199}$$

where $A' = \Sigma_{s=1}^{k-1}(\beta^2 + \omega_s^2)$, and B' is as defined before. We then have $P^+_{1j} = P^-_{1j}$, $j = 1,2,3$. For the example taken, $A' = 17$, and the value of ω_k satisfying (199) is 2.56 rad/sec.

For optimally locating the measurement $\omega = \omega_k$ so that P_{11} is minimized, equation

(185a) needs to be satisfied. This yields

$$-2\beta^2 \sum_{s=1}^{k} \alpha^2(\omega_s) + 3\omega_k^2 \sum_{s=1}^{k} \alpha^2(\omega_s) + \sum_{s=1}^{k} \omega_s^2 \alpha^2(\omega_s) = 0, \qquad (200)$$

which simplifies to

$$2\omega_k^4 + \omega_k^2(4\beta^2 + 3A') + 2A'\beta^2 - B\ni + 2\beta^2 = 0. \qquad (201)$$

For $\omega_4 \in [1,\infty)$ in our example, relation (194) gives a value of 1.148 rads/sec.

4.5. Extensions to some nonlinear sdof-systems.

In this section we extend the results obtained dealing with the parameter identification of a linear system to nonlinear systems that can be described by one degree of freedom. The proofs of all the results follow suit from those of the previous section and therefore have been omitted. Here we shall work directly in the time domain.

Consider a structure modelled by a nonlinear differential equation

$$m\ddot{x} + f(x, \dot{x}) = q(t), \qquad x(0) = \dot{x}(0) = 0, \qquad (202)$$

where within a certain range of response the nonlinear term can be approximated by

$$f(x,\dot{x}) = \sum_{n,m} \beta_{nm} x^n(t)\dot{x}^m(t). \tag{203}$$

The system is assumed to be nonhysteretic. Let us say that the response of the system is measured at times $t = t_k$, $k = 1,2,\ldots$, $t \in (0, T)$, and that the aim is to locate the instants $t = t_k$ when measurements should be made so that the data collected thereat would yield the minimal variance of any one of the parameters m or β_{mn} whose accurate identificaiton is required. We shall assume that x, $\dot{x}$ and $\ddot{x}$ and their derivatives are continuous functions of time for $t \in (0, T)$.

If the complete time histories x, $\dot{x}$ and $\ddot{x}$ were available (actually they are not), then equation (202) could be rewritten as

$$mz_1(t) + c_1z_2(t) + c_1z_3(t) + \cdots + c_Lz_{L+1}(t) = q(t), \tag{204}$$

where the z's correspond to the corresponding time functions. $L = m \times n$ and the coefficients β_{mn} are assembled into a one-dimensional array c. Identification of the parameter vector $\theta = [m\ c^T]^T$ under noisy measurement conditions would lead to the relations

$$q = H\theta + \varepsilon, \tag{205}$$

where

$$q = [q(t_1)\ q(t_2) \cdots q(t_N)]^T,$$

$$H = [z(t_1) z(t_2) \ldots z(t_{L+1})]^T, \quad z(t) = [z_1(t)\ z_2(t) \ldots z_{L+1}(t)]^T,$$

$$\varepsilon = \text{the zero-mean white measurement noise.} \tag{206}$$

The covariance of the estimate can be written, as before, as the $(1 + mn) \times (1 + mn)$ matrix

$$P = (H^T R^{-1} H)^{-1}, \tag{207}$$

where R is the noise covariance matrix and is taken to be diag $(\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2)$. Once again the extremal condition for P with respect to a measurement instant t_k can be expressed, as before, by the relation

$$\frac{\sigma_k^2}{2}\left(\frac{\partial P}{\partial t_k}\right)_{rr} = -\left[\sum_j P_{rj} z_j(t_k)\right]\left[\sum_s P_{sr} \dot{z}_s(t_k)\right] + \frac{\dot{\sigma}(t_k)}{\sigma(t_k)}\left(\sum_j P_{rj} z_j\right)^2 = 0. \tag{208}$$

where z_j is the j th element of z. (We note that this result follows directly from equation (165) if we set b to 0, a to z, and ω_k to t_k.) We then have the following result.

Theorem 5. For a given forcing function and any $r \in \{1,2,\ldots, mn + 1\}$, there may exist times t_k such that the inclusion of data at those times does not yield any improvement in the variance P_{rr} of our estimate of the r th parameter. Specifically, when for some $t_k \in (0,T)$

$$\sum_{j=1}^{mn=1} P_{rj} z_j(t_k) = 0 \tag{209}$$

is satisfied, then $t = t_k$ is such an instant.

Proof. The proof follows along the lines of Theorem 1. ■

Corollary 6. For values of t_k for which $\Sigma P_{rj} z_j (t_k) = 0$, the variance of the estimate of the r th variable as well as its cross-covariance is unaffected by the measruement at $t = t_k$.

Proof. As before, the condition implies that

$$P_{rj}^{+} = P_{rj}^{-}, \qquad j = 1,2,...mn + 1. \tag{210}$$

The proof follows, as did that of Corollary 1, from the proof of Theorem 5. ■

Theorem 6. The optimal measurement times t_k, $t_k \in (0, T)$, if they exist all, which minimize the variance P_{rr} satisfy the following relation

$$\sum_{s=1}^{mn+1} P_{sr} \dot{z}_s(t_k) = \frac{\dot{\sigma}(t_k)}{\sigma(t_k)} \sum_{j=1}^{mn+1} P_{rj} z_j . \tag{211}$$

Proof. The proof follows along the lines of Theorem 2.■

THEOREM 7. The measurement times $t_k \in (0, T)$ which cause

$$\sum_{i=1}^{\ell} P_{s_i s_j}, \quad \ell \leq mn + 1, \tag{212}$$

to be extremal are given by the relation

$$z^T(t_k)P\Lambda P\dot{z}(t_k) = \frac{\dot{\sigma}(t_k)}{\sigma(t_k)} z^T(t_k)P\Lambda Pz(t_k) \tag{213}$$

when Λ is the zero matrix whose n th diagonal element is 1 if $n = s_i$, $i = 1,\ldots, \ell$, $s_i \in (1,\ldots, mn + 1)$.

Proof. The proof is exactly along the lines of Theorem 4. ■

Theorem 8. If $z\,(t_k)$ is such that for a given set s_i, $i = 1, 2,\ldots, l$; $\ell \leq mn + 1$,

$$\Lambda Pz(t_k) = 0 \tag{214}$$

then there exist time $t_k \in (0,T)$ such that $\Sigma_{i=1}^{\ell} P_{s_i s_i}$ remains unaffected. In fact, $P_{r s_i}$, $r = 1, 2,\ldots, mn + 1$, $i = 1,2,\ldots, \ell$ remain unchanged by the new measurement at time t_k.

Proof. The proof follows along the same lines at that of Theorem 1. ■

4.6. Remarks and Conclusions.

In this section we have tried to understand the optimal measurement strategy for identifying the parameters of a single-degree-of-freedom dynamic system.

For a linear system, we have shown that, in general, data acquired at all frequencies, in the interval $\omega_k \in I_\Omega$, do not equally enhance our knowledge of the parameters being estimated. Specifically, one can often, given a data stream collected at frequencies ω_i, $i = 1,2,\ldots, k-1$, forecast the next frequency at which data collection would be maximally beneficial to obtaining a more confident estimate of any one of the desired parameters. Likewise, one can predict the frequency at which data collection would have no influence on improving the uncertainty in our estimate of any desired parameter. It is shown that for $\{\omega \in I_\Omega / f(\omega), g(\omega) \neq 0\}$, data at all the frequencies carry information about the damping parameter c. Also as opposed to the optimal measurement location ω_k for identification of m and k, which do depend on the previous measurement locations $\omega_1, \omega_2, \ldots \omega_{k-1}$, the optimal locations for the identification of c do not depend on the locations of the measurement stream. They are solely controlled by the nature of the forcing functions used in the identification procedure.

Irrespective of which parameter is being identified, the optimal measurement locations do not depend on the values of the parameters. It is noted that the solution of equations (185) and (186) may not exist for any ω belonging to the open interval I_Ω. For such situations the optimal locations would have to be chosen as the end point(s) of the interval. Also, it is observed that the optimal measurement locations do not depend on the system output x(t). Thus the optimal locations can be calculated a priori to obtaining the measurement stream. A numerical example has been included to illustrate the analytical results obtained. Furthermore, analytical results relating to minimization of one or some sum of the variances of the estimates are also provided.

The results are extended to nonhysteretic nonlinear oscillators when time data of displacement, velocity and acceleration can be obtained for a given time-dependent forcing function. It is shown that there may exist certain times t_k at which measurements provide no additional information regarding any one of the parameters desired to be most accurately estimated. Similarly there exist times t_k at which measurements provide maximal information about a desired parameter. The relations that these times t_k satisfy in each of the two cases have been analytically deduced.

The results of this section, it is hoped, will shed light on the manner in which experimentation can be performed so that the amount of data handling and reduction required could perhaps be significantly decreased in the dynamic testing of structural and mechanical systems.

SECTION 5. TRADEOFFS BETWEEN IDENTIFICATION AND CONTROL IN DYNAMIC SYSTEMS

5.1. INTRODUCTION

The control of large flexible structures is an area that has attracted considerable interest in recent years from both the professional and the research community [7], [21-24]. This interest is primarily motivated by the need to precisely control flexible structures in various developing fields of modern technology. In the area of earthquake engineering the reduction of structural response may be necessary to reduce internal stresses caused by dynamic loads thereby reducing the damage potential and increasing the useful life of the structure [25]. In space applications the availability of Space Shuttle to transport large payloads into orbit at reasonable costs presents an opportunity for large systems to perform new missions in space. However, because of launch weight and volume constraints these structures are generally very flexible and pose new challenges in all aspects of control such as attitude control and maneuvering, precision pointing, vibration attenuation and structural and shape control [7].

A necessary prelude to the effective control of a structure, is a knowledge of its characteristics and properties. In other words, one needs to have information about the structural system so that adequate control algorithms can be devised. This has led to a considerable interest in the identification of structures subjected to dynamic loads[8-10]. For structural systems that are described by parametric models, this involves knowledge of the nature of the governing differential

equations and knowledge of the values of the parameters that are involved, or at least knowledge of the bounds within which the parameters lie. Clearly, the better the system is identified (the smaller the bounding intervals within which the parameters are known to lie), the more finely tuned the controller can be made, so that for a given amount of available control energy, the control would be more efficient. The less knowledge we have about the structural system the more robust the controller needs to be, and, in general, the less efficient the control. Thus heuristically speaking there exists a duality between the concepts of identification and control, because 1) robust controllers may require reduced efforts at identification (for purposes of control), and, 2) increased efforts at identification may require less robust and more efficient controllers. However the tradeoffs between identification and control, from a practical standpoint, are still usually difficult to assess quantitatively. Little work has been reported to date in this area of quantitative cost-benefit analysis between these two dual concepts.

In this section we formulate the trade-off problem between identification and control, and study in a quantitative manner their duality through the use of the intermediary concept of an optimal input. Thus the section attempts to answer the following question: given that the optimal input time function is to have a certain prescribed energy, how does it change in character and in its effectiveness as one changes the objective criterion from one that emphasizes control to one that emphasizes identification? While the analytical work presented here has been motivated by our need to control flexible structural systems the methodology developed and the results obtained are applicable to all systems governed by ordinary differential equations; thus it applies to systems commonly met with in chemical, civil, electrical, and mechanical engineering. Some simple numerical examples are provided to indicate the quantitative nature of the results and provide a feel for them. The results in these examples show that significant trade-offs exist between identification

and control and that for the same amount of energy in the input signal, the emphasis on control could lead to very high covariances of the parameter estimates. Similarly inputs that are optimal for identification could yield responses whose mean squared values may be several times those obtained for inputs that yield optimal control.

5.2. PROBLEM FORMULATION

Consider a dynamic system modelled by the first order set of differential equations

$$\dot{x}(t) = F_1 x + G_1 f \tag{215}$$

$$z(t) = H_1 x(t) + v(t) \tag{216}$$

where x is an nX1 state vector, f is an mX1 control vector, z is an rX1 measurement vector and the nX1 initial condition vector, x_0, is given. We shall assume that the measurement noise is representable as a zero mean Gaussian White Noise process so that

$$E[\, v(t) \,] = 0, \text{ and,} \tag{217}$$

$$E[\, v(t)\, v(\tau) \,] = R_1\, \delta\, (\, t - \tau\,)\, . \tag{218}$$

Let the vector of unknown parameters in the system modelled by equations (215) and (216) be given by the pX1 vector θ. Let us assume that the identification is carried out with an efficient unbiased estimator so that the covariance of the estimate of θ namely, $\hat{\theta}$ is provided by the inverse of the Fisher Information Matrix[13]. Hence,

$$\mathrm{Cov}[\, \hat{\theta} \,] = M^{-1}. \tag{219}$$

The matrices F_1 and G_1 are taken to be functions of, in general, the parameter vector θ. The optimal input for identification of the parameter vector θ is then sought such that a suitable norm related to the matrix M is maximized or minimized. Different measures of performance related to M have been used in the literature [19]:

1. A - Optimality, where $Tr(M^{-1})$ is minimized;
2. D - Optimality, where the determinant of M^{-1} is minimized; and,
3. E - Optimality, where the maximum eigenvalue of M^{-1} is minimized.

In this section, for expository purposes, we shall use the criterion for obtaining the optimal inputs for identification as the maximization of the $\text{Trace}\{W^{1/2} M W^{1/2}\}$ where W is a suitable positive definite weighting matrix. Thus the criterion for obtaining the optimal input for parameter identification is taken to be

$$J_I = \int_0^T \text{Trace}\{ \psi_p^T H_1^T R_1^{-1} H_1 \psi_p \} \, dt \qquad (220)$$

where

$$\psi_p = X_p W^{1/2}, \qquad (221)$$

and the matrix X_p is given by,

$$[X_p]_{i,j} = \frac{\partial x_i}{\partial \theta_j}. \qquad (222)$$

In addition to the objective function generated by our need for identification, the objective function required to be maximized for control is,

$$J_C = -\int_0^T \{ x^T Q_1 x \} \, dt . \qquad (223)$$

Here Q_1 is a symmetric positive definite, nXn weighting matrix. This then yields the composite objective function which is required to be maximized as

$$J = -(\alpha/2) \int_0^T \{x^T Q_1 x\} \, dt + (\beta/2) \int_0^T \mathrm{Trace} \{ \psi_p^T H_1^T R_1^{-1} H_1 \psi_p \} , \qquad (224)$$

where a nd β are positive scalars. Clearly, when $\alpha >> \beta$, finding f(t) to maximize J is tantamount to finding the optimal control for the system (215)-(216), while when $\beta >> \alpha$, the f(t) that maximizes J is simply the optimal input for identification of the pX1 parameter vector θ. In particular, when $\alpha = 0$, and $\beta = 1$, the optimal input for 'best' identification is obtained; when $\alpha = 1$, and $\beta = 0$, the optimal input for 'best' control is obtained. Denoting the nX1 vector

$$x_{\theta_i} = \frac{\partial x}{\partial \theta_i} \qquad (225)$$

and assuming that the matrix W is diagonal, so that,

$$W = \mathrm{Diag}(w_1, w_2, . \; . \; . \; . \; . , w_p) \qquad (226)$$

we can generate an augmented n(p + 1) vector,

$$y(t) = \begin{bmatrix} x(t) \\ w_1^{1/2}\,\partial x(t)/\partial\theta_1 \\ w_2^{1/2}\,\partial x(t)/\partial\theta_2 \\ \cdot \quad\quad \cdot \\ w_p^{1/2}\,\partial x(t)/\partial\theta_p \end{bmatrix}, \text{ with, } \quad y(0) = \begin{bmatrix} x_0 \\ 0 \\ 0 \\ \cdot \\ 0 \end{bmatrix} \tag{227}$$

which is then governed by the differential equation

$$\dot{y} = F\,y + G\,f\,,\ y^T(0) = \{\, x_0^T\ ,\ 0 \,\} \tag{228}$$

where F is the n(p + 1) X n(p + 1) matrix given by

$$F = \begin{bmatrix} F_1 & \cdot & \cdot & \cdot & \cdot \\ w_1^{1/2}F_{\theta_1} & F_1 & \cdot & \cdot & \cdot \\ w_2^{1/2}F_{\theta_2} & \cdot & F_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ w_p^{1/2}F_{\theta_p} & \cdot & \cdot & \cdot & F_1 \end{bmatrix}, \tag{229}$$

and,

$$G = \begin{bmatrix} G_1 \\ w_1^{1/2} G_{\theta_1} \\ w_2^{1/2} G_{\theta_2} \\ \cdot \quad \cdot \\ w_p^{1/2} G_{\theta_p} \end{bmatrix} , \tag{230}$$

with

$$F_{\theta_i} = \frac{\partial F_1}{\partial \theta_i} , \quad G_{\theta_i} = \frac{\partial G_1}{\partial \theta_i} , \quad i = 1,2, \ldots, p. \tag{231}$$

We note in passing that the stability of the equation set (228) is controlled by the stability of the equation set (215) since the eigenvalues of the matrix F are identical to those of the matrix F_1, as seen in equation (229), except for the increased multiplicities. The objective function (224) can now be rewritten, after some algebra, as

$$J = -(\alpha/2) \int_0^T y^T Q\, y^T \, dt + (\beta/2) \int_0^T y^T H^T R^{-1} H\, y \, dt \tag{232}$$

where, the matrices Q, H, and R^{-1} are the block diagonal matrices given by

$$\begin{aligned} Q &= \mathrm{Diag}\{ Q_1, O, O, \ldots, O, O\}, \\ H &= \mathrm{Diag}\{O, H_1, H_1, H_1, \ldots, H_1\}, \text{ and,} \\ R^{-1} &= \mathrm{Diag}\{ R_1^{-1}, R_1^{-1}, R_1^{-1}, \ldots, R_1^{-1} \} \end{aligned} \tag{233}$$

Thus the objective function needs to be maximized under the constraint equations (228) and the

energy constraint

$$\int_0^T f^T f \ dt = E, \tag{234}$$

where the parameter E is given a priori.

5.3. Determination of Optimal Inputs for Simultaneous Identification and Control

Having formulated the problem for constrained maximization, we next use the standard Lagrange multiplier method to obtain the function f(t) which maximizes the objective (232) subject to (228) and (234). Using the Lagrange multipliers $\lambda(t)$ and $\nu(t)$ we therefore obtain the augmented objective function to be

$$\bar{J} = -\frac{\alpha}{2}\int_0^T \{y^T Q\, y\}\ dt + \frac{\beta}{2}\int_0^T \{y^T H^T R^{-1} H\, y\}\ dt + \int_0^T \lambda^T(t)\{\, Fy + Gf - \dot{y}\}dt + \int_0^T \frac{\eta(t)}{2}(f^T f - \dot{y}_{N+1})\ dt \tag{235a}$$

Here we augmented the state vector by the variable

$$y_{N+1}(t) = \int_0^t f^T(\tau)\, f(\tau)\, d\tau \tag{235b}$$

where N = n(p+1) and we use the additional Lagrange multiplier $\eta(t)$. By doing this we can satisfy the equality constraint (234) without having to resort to the usual trial and error procedure.

Taking the first variation, we obtain the following set of equations for the variables $y(t)$, $\lambda(t)$ and $\eta(t)$ (see Appendix C):

$$\dot{y}(t) - F y = \frac{1}{\eta} V \lambda(t), \quad y(0) = y_0$$

$$\dot{\lambda}(t) + F^T \lambda(t) = -\alpha \{Q\} y + \beta \{H^T R^{-1} H\} y, \quad \lambda(T) = 0, \tag{236}$$

$$\dot{y}_{N+1} = f^T f = \frac{1}{\eta^2} \lambda^T(t) V \lambda(t), \quad y_{N+1}(0) = 0, \quad y_{N+1}(T) = E$$

$$\dot{\eta}(t) = 0$$

where $V = (GG^T)$. We note that the equation set (236) constitutes a nonlinear two point boundary value problem containing $2[np + n+ 1]$ first-order differential equations. The optimal input vector, $f(t)$, is obtained through the solution of this two point boundary value problem using the relation:

$$f(t) = -\frac{1}{\eta} G^T \lambda(t) \tag{237}$$

It is interesting to note that had we used the objective function

$$J_C' = \left[\int_0^T x^T Q_1 x \, dt \right]^{-1} \tag{238}$$

instead of J_C in equation (223) we would obtain,

$$J' = \frac{\alpha}{2} J_C' + \frac{\beta'}{2} J_I \tag{239}$$

where J_I is given in equation (220), and the relative weightings of the contributions of the control and the identification objectives are denoted by α and β'. Following the same procedure as before and using the augmented vector y, maximization of this objective function, J', along with the constraints (228) and (234) would then yield the following set of equations:

$$\dot{y}(t) - F\,y(t) = \frac{1}{\bar{\tilde{\eta}}}\, V\, \tilde{\lambda}(t)\,, \quad y(0) = y_0\,,$$

$$\dot{\tilde{\lambda}}(t) + F^T\,\tilde{\lambda}(t) = -\alpha\,\{Q\}y + (\beta'\Delta)\,\{H^T R^{-1} H\}y\,, \quad \tilde{\lambda}(T) = 0\,, \tag{240}$$

$$\dot{y}_{N+1} = \frac{1}{\tilde{\eta}^2}\tilde{\lambda}^T(t)\,V\,\tilde{\lambda}(t), \quad y_{N+1}(0) = 0, \quad y_{N+1}(T) = E$$

$$\dot{\tilde{\eta}}(t) = 0\,,$$

where, Δ is the positive quantity defined by

$$\Delta = [\int_0^T y^T Q\, y\, dt\,]^2 = [\int_0^T x^T Q_1 x\, dt\,]^2 \tag{241}$$

The optimal input is obtained from the relation

$$f(t) = -\frac{1}{\bar{\tilde{\eta}}}\, G^T\, \tilde{\lambda}(t)\,. \tag{242}$$

Comparing equations (236) and (240) we observe that the only difference that arises in the use of the objective function (239) instead of (232) is the effective change in weighting parameter β. Setting

$\beta'\Delta = \beta$ the equation set (240) becomes identical with the set (236). The lagrange multipliers $\lambda(t)$ of (236) and $\tilde{\lambda}$ of (240) are related by $\tilde{\lambda} = \lambda\Delta$; similarly, $\tilde{\eta} = \eta\Delta$. It should be noted that the two equations become identical only when the function y(t) in (241) corresponds to the response for the optimal input f(t), i.e., the solution, y, of the set (236). With this rescaling of the parameter β', the optimal input f(t) is identical for the two objective functions J and J' of (224) and (238). Having thus shown the quasi-equivalence of the two objective functions (224) and (239) through this rescaling, in this sequel, we shall illustrate our results by using the objective function in the form of equation (224) which leads to the boundary problem described in equation (236).

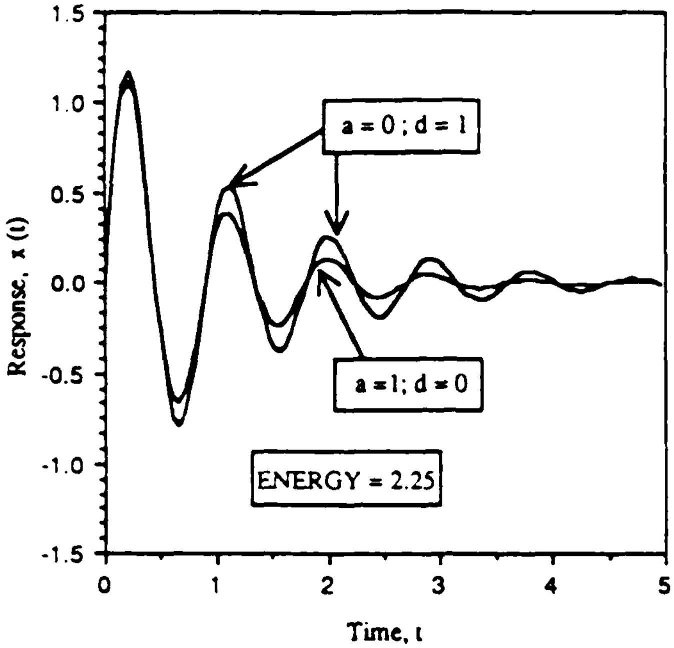
This two-point boundary value problem can be numerically solved in various ways. An extensive literature on numerical techniques for solving such problems is available[26]. Among the methods most commonly used are multiple shooting techniques[26] with Newton-Raphson iterations, and the Kalaba Method[27] where the two point boundary value problem is converted to an equivalent Cauchy initial value problem. In this sequel, the equation set (236) is solved using the multiple shooting technique with Newton iteration.

5.4. Illustrative Example

To exemplify the concepts developed, let us consider a system modelled by a single degree-of-freedom oscillator described by the differential equations

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)\,; \quad x_1(0) = a_0\,,\ x_2(0) = b_0 \qquad (243)$$

where f(t) is the optimal input to be applied. Denoting $x = [x_1 \;\; x_2]^T$, $x_k = \partial x/\partial k$, and $x_d = \partial x/\partial c$, the objective function is taken to be



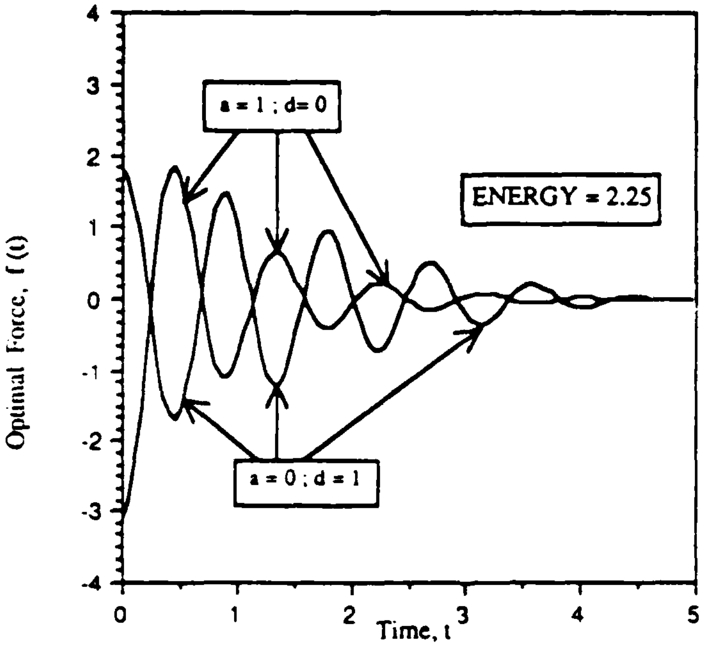


Figure 22A. <u>Comparison of Optimal Force-Time Histories for Identification & Control</u>.

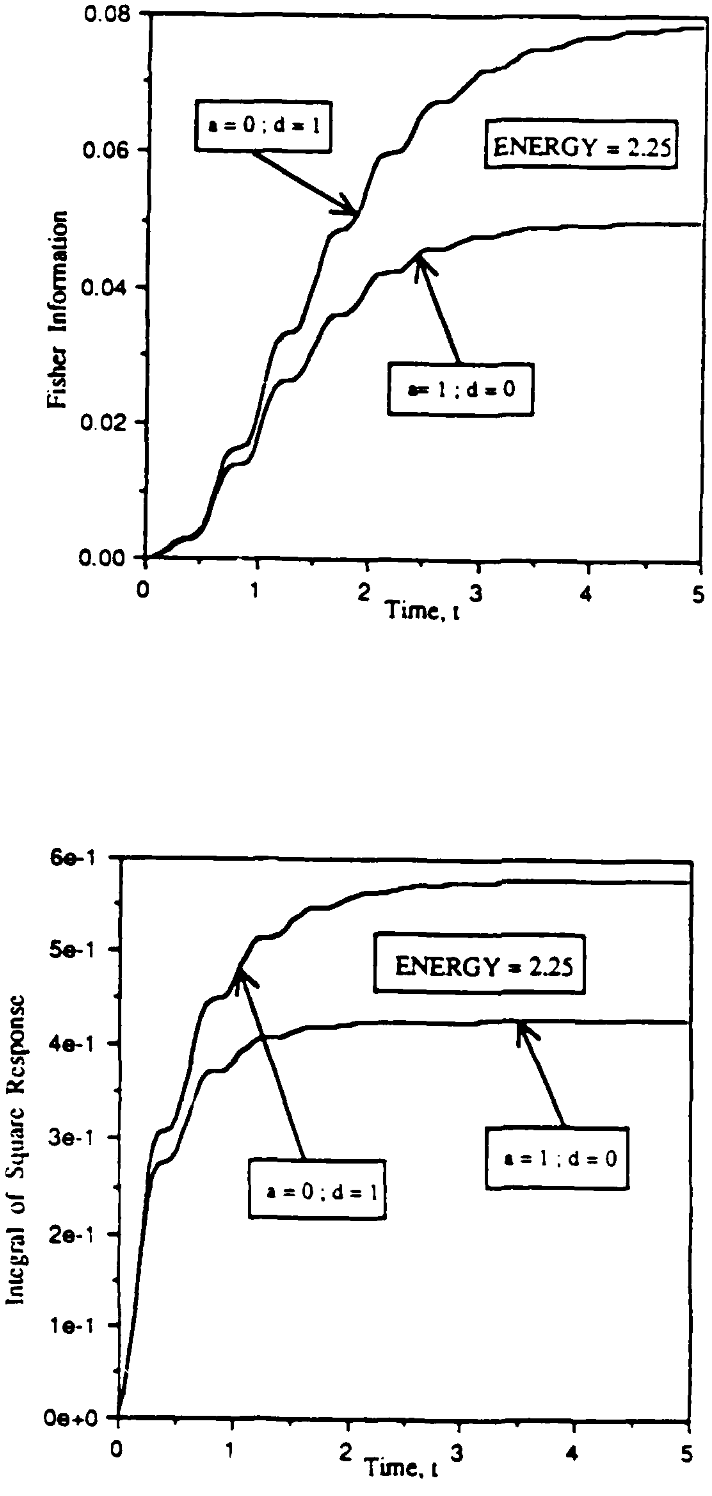


Figure 22B. Comparison of Fisher Information and RMS Response to Optimal Force-Time Histories for Identification & Control.

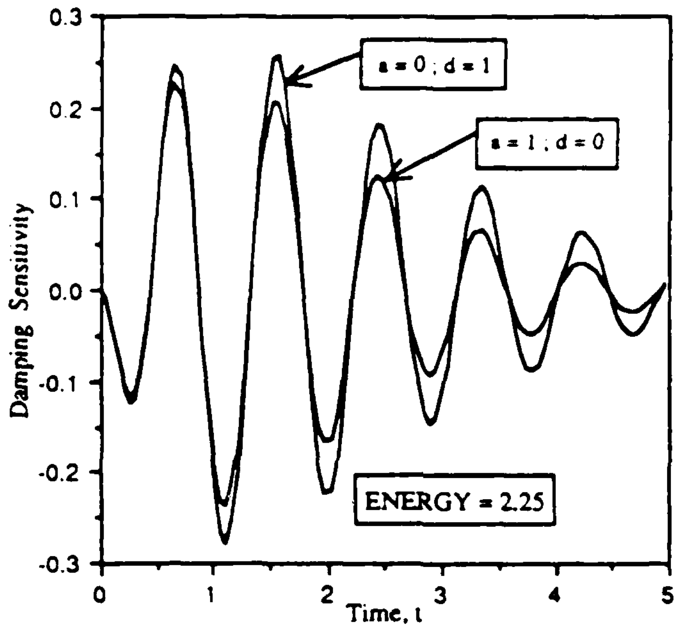


Figure 22C. Comparison of Damping Sensitivity to Optimal Force-Time Histories for Identification and Control.

$$J(T) = -a\,J_c(T) + b\,J_k(T) + d\,J_d(T) \tag{244}$$

where,

$$J_c(T) = \int_0^T x_1^2 \; dt \; , \tag{245}$$

$$J_k(T) = \int_0^T {x_k}^2 \; dt \; , \tag{246}$$

and,

$$J_d(T) = \int_0^T {x_d}^2 \; dt \; , \tag{247}$$

The weighting factors a, b, and d are taken to be non-negative. This may be thought of as being produced by choosing $\alpha = a$, $\beta = 1$, $W = \mathrm{Diag}\,(b\ ,d\ )$, $Q_1 = \mathrm{Diag}(1, 0)$ and $H_1 = [1\ \ 0]$ and the scalar $R_1 = \sigma^2 = 1$ in our general formulation of equations (224) - (232). The six component vector y is then given by

$$y = \begin{bmatrix} x \\ b^{1/2} x_k \\ d^{1/2} x_d \end{bmatrix} , \tag{248}$$

and the matrix F becomes

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -k & -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -b^{1/2} & 0 & -k & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -d^{1/2} & 0 & 0 & -k & -c \end{bmatrix}. \qquad (249)$$

Since the vector G = [0 1 0 0 0 0], the vector $\{V\lambda/\eta\}$ has only one non-zero component namely, $\lambda_2(t)/\eta$, where $\lambda_2(t)$ is the second element of the vector $\lambda(t)$. Figure 22 shows some of the numerical results for the following parameters values (which we shall assume are taken in consistent units):

$$b=0,\ T=5,\ E = 2.25;\ \text{and}, \qquad (250)$$

$$k=50,\ c=2,\ x_1(0) = 0,\ x_2(0) = 10. \qquad (251)$$

These parameters thus look at a single degree-of-freedom oscillator whose spring constant is 50 units, and whose viscous damping is 2 units. This yields a system which has an undamped natural frequency of vibration of about 7 radians/sec and a percentage of critical damping of about 15%. It is subjected to an initial velocity of 10 units. The aim is to study the trade-off between 1) identifying (in equation (243)) the damping parameter, c, in the best possible way, and 2) controlling the system so that its mean square response over the time period T is a minimum, given that an input (forcing function) of 5 units duration with an energy of up to 2.25 units is to be used. The two point boundary value problem posed in equation set (236) is numerically solved using the standard multiple shooting technique[15]. The local error tolerence during integration of the differential equations and the permitted error in the satisfaction of the boundary conditions are each set to 10^{-4}. The responses of the system together with the optimal inputs as obtained from equation (237) are shown for the two extreme cases: 1) a=0, d=1, corresponding to the optimal input

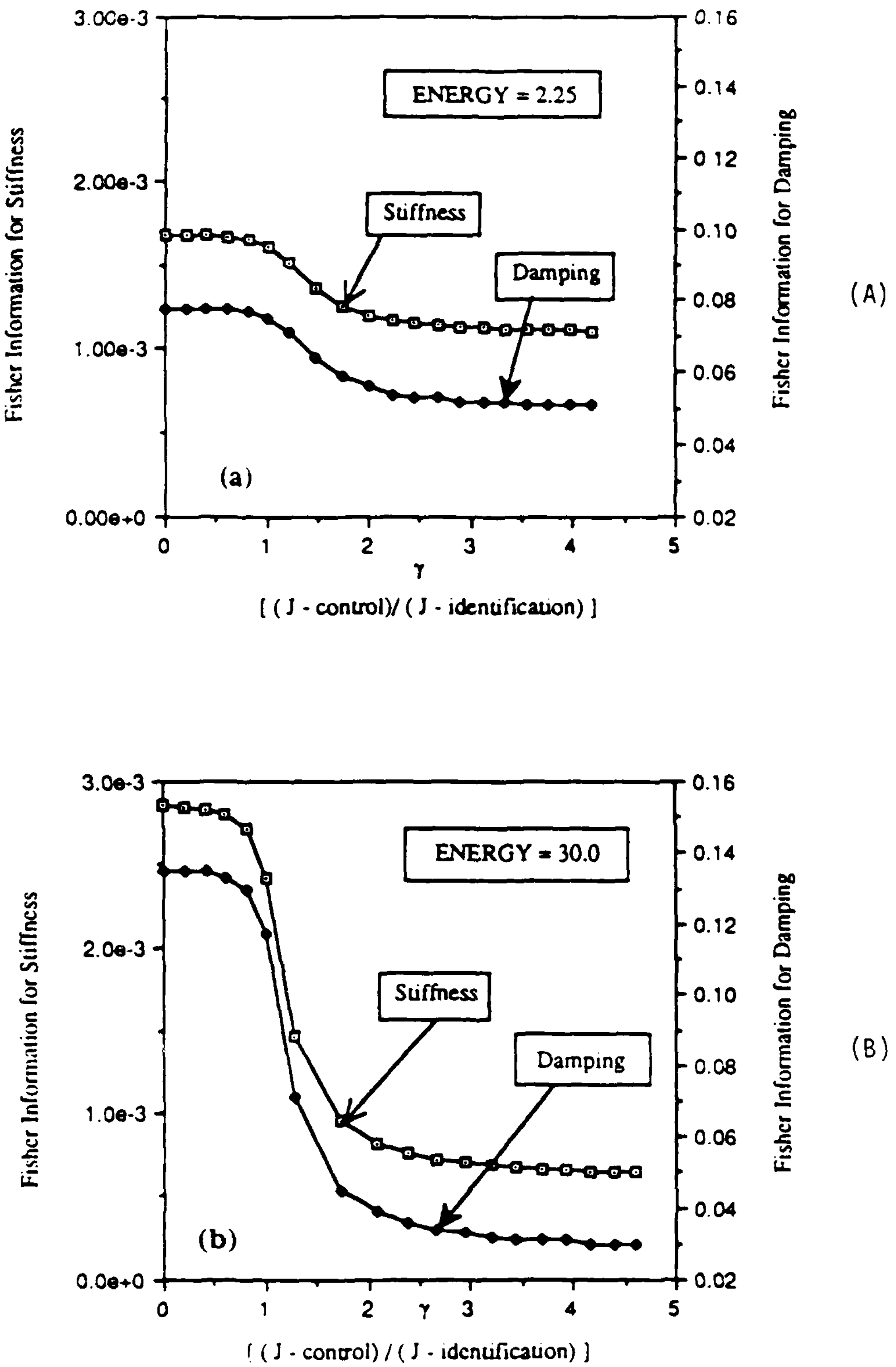


Figure 23. Comparison of Fisher Matrices for Stiffness and Damping.

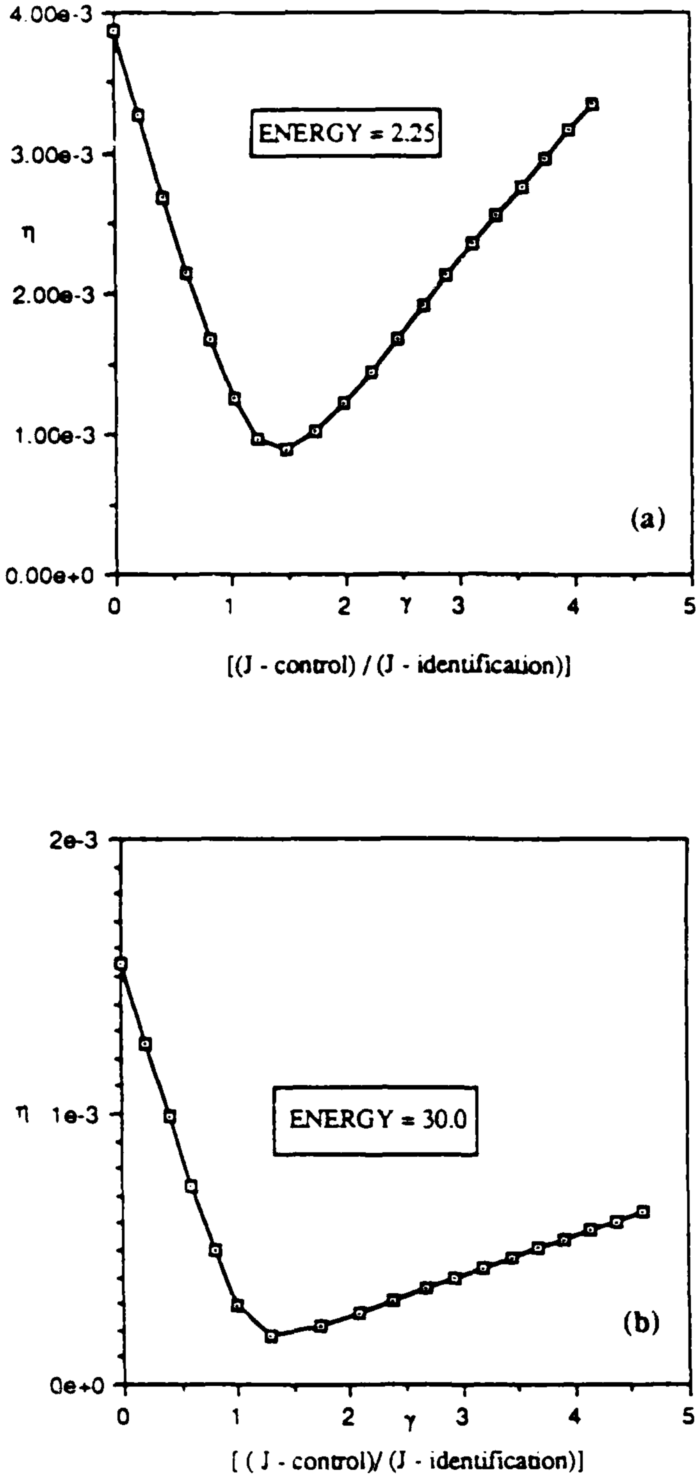


Figure 24. Variation of η with different J_C/J_I.

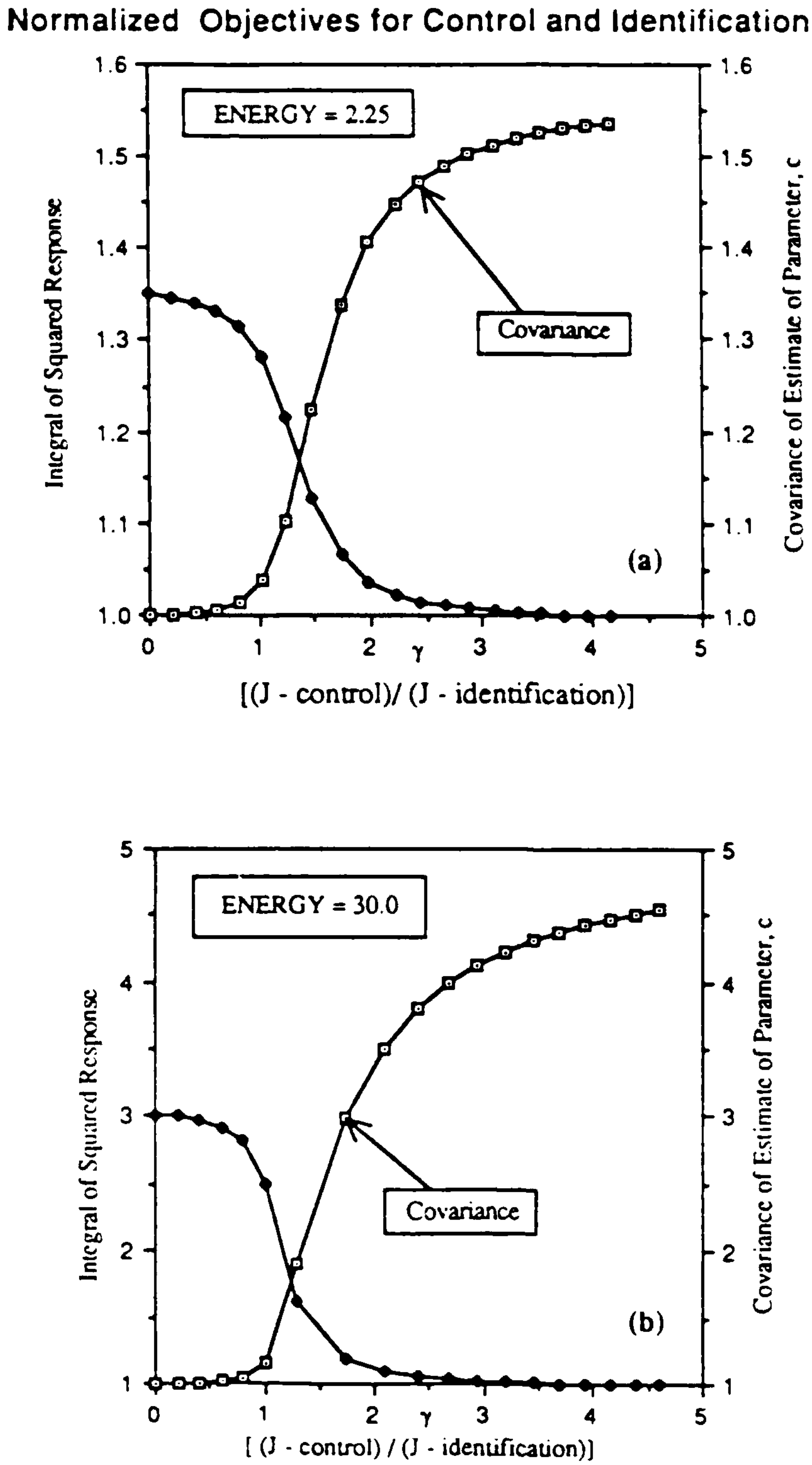


Figure 25. Tradeoffs Between Identification and Control for Different Energy Levels.

requireu for identification of the damping parameter c, and, 2) a=1, d=0, corresponding to the optimal input required for minimizing the response. As seen from Figure 22(B), the optimal inputs required for 'best' identification and for 'best' control (the term 'best' is used in in terms of the cost function (244) utilized) are widely different form each other. In fact they are seen to be, for the entire duration over which they last, almost exactly out of phase. Differences in the response of the system to combined influence of the initial velocity and the forcing functions obtained for the two cases are shown in Figure 22(A). Figure 22(C) shows the Fisher Information Matrices for damping, which in this case are scalars, namely $J_d(t)$, for the abovementioned two extreme cases, as a function of time, t. The difference between these at T= 5 is about 55%. Alternatively put, the input forcing function, which controls the system response maximally, causes a response which is only about 55% as informative about the system parameter c as that caused by a forcing function that is designed to maximally provide information about the parameter, c. The manner in which the integral of the response quantity squared, $J_c(t)$, changes with time for the two cases mentioned above is shown in Figure 22(D). As seen, at T = 5, the optimal control input is about 35% more effective in reducing the mean square response than the input which optimally determines the parameter c. Figure 22(E) provides the sensitivity of the response to the damping parameter (at c=2) as a function of time.

Figure 23(A) shows the manner in which the Fisher Information matrices $J_d(T)$ and $J_k(T)$ change for various values of the ratio, $\gamma = [\{a J_c(T)\}/\{d J_d(T)\}]$ when the fisher values are normalized to unity. It is to be noted that the optimal input when a=0 corresponds to the that required for 'best' estimation of the parameter c in equation (243). Figure 23(B) shows the effect of changing the available control energy, E, from a value of 2.25 to 30 keeping all other parameters the same. From a loss of information in the parameter c of 55% in the case of E=2.25, the loss in information when E = 30 jumps to about 450%. Similarly, the extent to which the system's performance can be controlled deteriorates by about a factor of 3 if one aims at purely identification instead of control. The parameter η which is calculated for each objective function ratio, γ, is shown in Figure 24. As mentioned in the formulation, this quantity is simultaneously solved for, in the set (236), thereby eliminating the need to find its value by trial and error. Had this not been done a very high computational expense would have been incurred to ensure that the energy

constraint is satisfied. Noting that the inverse of $J_d(T)$, for an efficient unbiased estimator, is the covariance of the estimate of the parameter c, Figures 25(A) and 25(B) provide the trade off between control and identification. As seen in Figure 25(A), in going from $\gamma = 0$ to $\gamma = 4$, $J_c(T)$, the mean square response, **falls off** by about 35%; similarly the covariance of the estimate of c **increases** by about 55% as γ varies over the same interval. For larger values of the input energy, Figure 25(B) shows that significant reductions in J_c and significant increases in the covariance of the parameter estimates can occur.

5.5. Conclusions

In this section we have presented an approach to quantifying the tradeoff between the tasks of control low development and plant identification. To the best of our knowledge this tradeoff has never been analyzed quantitatively before. The problem is formulated in the context of optimal control and optimal identification through the intermediary concept of an optimal input. A suitable objective function is chosen so that the emphasis from control to identification can be changed in a continuous manner. It is shown that the duality between identification and control can be quantified by determining optimal inputs, which have a specified amount of energy, and which minimize the objective function. Augmenting the state by an additional variable allows a simultaneous solution of the optimization problem together with the energy constraint. Using variational calculus this leads to a two point boundary value problem that is nonlinear due to the introduction of the energy constraint. The boundary value problem is solved numerically using the multiple shooting technique and Newton-Raphson iterations.

A numerical example, which deals with control and identification of the parameters of a single degree-of-freedom oscillator, is used to illustrate the concepts involved. It is shown that improved control leads to serious deterioration in the covariance of the parameter estimation and vice-versa. In general, as the energy of the input increases the tradeoffs between identification and control are shown to become more and more intense.

The above example shows the potential of the approach introduced in this section. The

numerical computations can be easily generalized to vibratory systems with many degrees of freedom, making the method presented here useful in studies of large flexible structures.

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APPENDIX A

If in the equation (101) $f(t) = \delta(t)$, the expressions for $Q_1(T)$ and $Q_2(T)$ when estimating A ($\alpha = 0$) may be written as:

$$Q_1(T) = p_1^2 a_1 + q_1^2 b_1 + p_2^2 a_2 + q_2^2 b_2 + 2p_1q_1c_1 + 2p_1p_2d + p_1q_2g$$
$$+p_2q_1h+2q_1q_2\ell+2p_2q_2c_2 \tag{A.1}$$

and

$$Q_2(T) = p'_1{}^2a_1+q'_1{}^2b_1+p'_2{}^2a_2+q'_2{}^2b_2+{}^2p'_1q'_1c_1+{}^2p'_1p'_2d$$
$$+p'_1{}^2q'_2{}^2g+p'_2q'_1h+2q'_1q'_2\ell+2p'_2q'_2c_2$$

(A.2)

where

$$a_i = \frac{T}{2} - \frac{\mathrm{Sin}(2\omega_i T)}{4\omega_i}, \tag{A.3}$$

$$b_i = \frac{T^3}{6} + \left(\frac{T^2}{4\omega_i} - \frac{1}{8\omega_i^3}\right)\mathrm{Sin}(2\omega_i T) + \frac{T}{4\omega_i^2}\mathrm{Cos}(2\omega_i T), \tag{A.4}$$

$$c_i = \frac{\mathrm{Sin}(2\omega_i T)}{8\omega_i^2} - \frac{T\mathrm{Cos}(2\omega_i T)}{4\omega_i}, \text{ and } i \in [1,2] \tag{A.5}$$

also

$$\omega_{1,2} \triangleq \lambda_{1,2} = \frac{k}{2Am}\left[(A+B+1) \pm \sqrt{(A+B+1)^2 - 4AB}\right. \tag{A.6}$$

Let us define

$$\omega_N = \omega_1 - \omega_2 \tag{A.7}$$

and

$$\omega_P = \omega_1 - \omega_2 \tag{A.8}$$

Hence

$$d = \frac{\mathrm{Sin}\,(\omega_N T)}{\omega_N} - \frac{\mathrm{Sin}\,(\omega_P T)}{\omega_P}, \tag{A.9}$$

$$g = \frac{\mathrm{Sin}(\omega_N T)}{\omega_N^2} - \frac{T\mathrm{Cos}(\omega_N T)}{\omega_N} + \frac{\mathrm{Sin}(\omega_P T)}{\omega_P^2} - \frac{T\mathrm{Cos}(\omega_P T)}{\omega_P}, \tag{A.10}$$

$$h = -\frac{\mathrm{Sin}\,(\omega_N T)}{\omega_N^2} + \frac{T\mathrm{Cos}\,(\omega_N T)}{\omega_N} + \frac{\mathrm{Sin}\,(\omega_P T)}{\omega_P^2} - \frac{T\mathrm{Cos}\,(\omega_P T)}{\omega_P}, \tag{A.11}$$

and

$$\ell = \left[\frac{T\,\mathrm{Cos}(\omega_P T)}{\omega_P^2} + \frac{1}{2}\left(\frac{T^2}{\omega_P} - \frac{2}{\omega_P^3}\right)\mathrm{Sin}(\omega_P T) + \frac{T\,\mathrm{Cos}(\omega_N T)}{\omega_N^2}\right.$$

$$\left. + \frac{1}{2}\left(\frac{T^2}{\omega_N} - \frac{2}{\omega_N^3}\right)\mathrm{Sin}(\omega_N T)\right] \tag{A.12}$$

Let us further define

$$D = k/(k - \lambda_1 m), \tag{A.13}$$

$$E = k/(k - \lambda_2 m), \text{ and} \tag{A.14}$$

$$ROOT = [(A+B+1)^2 - 4AB]^{1/2} \tag{A.15}$$

Therefore,

$$p_1 = \Bigg\{ \frac{-1}{m\omega_1}\left[\frac{1}{(A+D^2)^2} + \frac{1}{(A+E^2)^2}\right] - \frac{D^3}{Am\omega_1(A+D^2)^2}\left[\frac{A-B+1}{ROOT} - \frac{B+1+ROOT}{A}\right]$$

$$+ \frac{E^3}{Am\omega_1(A+E^2)^2}\left[\frac{A-B+1}{ROOT} + \frac{B+1-ROOT}{A}\right] - \frac{k}{4Am^2\omega_1^3} \cdot$$

$$\cdot \left[\frac{A-B+1}{ROOT} - \frac{B+1+ROOT}{A}\right]\left[\frac{1}{A+D^2} + \frac{1}{A+E^2}\right]\Bigg\} , \tag{A.16}$$

$$q_1 = \Bigg\{\frac{k}{4Am^2\omega_1^2}\left[\frac{A-B+1}{ROOT} - \frac{B+1+ROOT}{A}\right]\left[\frac{1}{A+D^2} + \frac{1}{A+E^2}\right], \tag{A.17}$$

$$p_2 = \Bigg\{ \frac{-1}{m\omega_2}\left[\frac{D}{(A+D^2)^2} + \frac{D}{(A+E^2)^2}\right] + \frac{D^2(A-D^2)}{2Am\omega_2(A+D^2)^2}\left[\frac{A-B+1}{ROOT} - \right.$$

$$\left. \frac{B+1+ROOT}{A}\right] - \frac{E^2(A-E^2)}{2A\omega_2(A+E^2)^2}\left[\frac{A-B+1}{ROOT} + \frac{B+1-ROOT}{A}\right] + \frac{k}{4Am^2\omega_2^3} \cdot$$

$$\cdot\left[\frac{A-B+1}{ROOT}+\frac{B+1-ROOT}{A}\right]\left[\frac{D}{A+D^2}+\frac{E}{A+E^2}\right] , \tag{A.18}$$

$$\dot{q}_2=\left\{\frac{-k}{4Am^2\omega_2^2}\left[\frac{A-B+1}{ROOT}+\frac{B+1-ROOT}{A}\right]\left[\frac{D}{A+D^2}+\frac{E}{A+E^2}\right]\right\} , \tag{A.19}$$

$$\dot{p}_1=\left\{\frac{-1}{m\omega_1}\left[\frac{D}{(A+D^2)^2}+\frac{E}{(A+E^2)^2}\right]+\frac{D^2(A-D^2)}{2Am\omega_1(A+D^2)^2}\left[\frac{A-B+1}{ROOT}-\frac{B+1+ROOT}{A}\right]\right.$$

$$-\frac{E^2(A-E^2)}{2Am\omega_1(A+E^2)^2}\left[\frac{A-B+1}{ROOT}+\frac{B+1-ROOT}{A}\right]-\frac{k}{4Am^2\omega_1^3}$$

$$\left.\cdot\left[\frac{A-B+1}{ROOT}-\frac{B+1+ROOT}{A}\right]\left[\frac{D}{A+D^2}+\frac{E}{A+E^2}\right]\right. , \tag{A.20}$$

$$\dot{q}_1=\left\{\frac{k}{4Am^2\omega_1^2}\left[\frac{A-B+1}{ROOT}-\frac{B+1+ROOT}{A}\right]\left[\frac{D}{A+D^2}+\frac{E}{A+E^2}\right]\right\} , \tag{A.21}$$

$$\dot{p}_2=\left\{\frac{-1}{m\omega_2}\left[\frac{D^2}{(A+D^2)^2}+\frac{E^2}{(A+E^2)^2}\right]+\frac{D^3}{m\omega_2(A+D^2)^2}\left[\frac{A-B+1}{ROOT}-\right.\right.$$

$$\left.\frac{B+1+ROOT}{A}\right] - \frac{E^3}{m\omega_2(A+E^2)^2}\left[\frac{A-B+1}{ROOT}+\frac{B+1-ROOT}{A}\right]+\frac{k}{4Am^2\omega_2^3}\ .$$

$$\cdot\left[\frac{A-B+1}{ROOT}+\frac{B+1-ROOT}{A}\right]\left[\frac{D^2}{A+D^2}+\frac{E^2}{A+E^2}\right]\ , \tag{A.22}$$

and

$$\dot{q}_2 = \left\{\frac{-k}{4Am^2\omega_2^2}\left[\frac{A-B+1}{ROOT}+\frac{B+1-ROOT}{A}\right]\left[\frac{D^2}{A+D^2}+\frac{E^2}{A+E^2}\right]\right\}\ . \tag{A.23}$$

Substituing equations (A.3 - A.23) into equations (A.1) and (A.2) yields the exact form of $Q_1(T)$ and $Q_2(T)$ for the solution of the OSLP for determining the parameter A.

Similarly if one seeks the solution of the OSLP for determining the parameter B (under the same impulsive base input) for the system governed by equation (101) and ($\alpha = 0$), one may write, after some algebraic manipulations:

$$\begin{aligned} Q_1(T) = {} & u_1^2 a_1 + v_1^2 b_1 + u_2^2 a_2 + v_2^2 b_2 + 2u_1v_1c_1 + 2u_1u_2d + u_1u_2g \\ & + u_2v_1h + 2v_1v_2\ell + 2u_2v_2c_2, \end{aligned} \tag{A.24}$$

and

$$\begin{aligned} Q_2(T) = {} & u_1'^2a_1 + v_1'^2b_1 + u_2'^2a_2 + v_2'^2b_2 + 2u_1'v_1c_1 + 2u_1'u_2'd \\ & + u_1'u_2'g + u_2'v_1'h + 2v_1'v_2'\ell + 2u_2'v_2'c_2 \end{aligned} \tag{A.25}$$

where a_i, b_i, and c_i ($i \in [1, 2]$) are given by equations (A.3 - A.6); and d, g, h and ℓ are given by equations (A.9 - A.12). Also

$$u_1 = \left\{\left[\frac{-D^3}{Am\omega_1(A+D^2)^2} - \left(\frac{k}{4Am^2\omega_1^3}\right)\left(\frac{1}{A+D^2} + \frac{1}{A+E^2}\right)\right]\left[1 + \frac{B-A+1}{ROOT}\right]\right.$$

$$\left. - \left(\frac{E^3}{Am\omega_1(A+E^2)^2}\right)\left[1 - \frac{B-A+1}{ROOT}\right]\right\} \quad , \tag{A.26}$$

$$v_1 = \frac{k}{4Am^2\omega_1^2}\left(\frac{1}{A+D^2} + \frac{1}{A+E^2}\right)\left[1 + \frac{B-A+1}{ROOT}\right] , \tag{A.27}$$

$$u_2 = \left\{\left[\frac{E^2(A-E^2)}{2Am\omega_2(A+E^2)^2} - \left(\frac{k}{2Am^2\omega_2^3}\right)\left(\frac{D}{A+D^2} + \frac{E}{A+E^2}\right)\right]\left[1 - \frac{B-A+1}{ROOT}\right]\right.$$

$$\left. + \left(\frac{D^2(A-D^2)}{2Am\omega_2(A+D^2)^2}\right)\left[1 + \frac{B-A+1}{ROOT}\right]\right\} \quad , \tag{A.28}$$

$$v_2 = \frac{k}{2Am^2\omega_2^2}\left(\frac{D}{A+D^2} + \frac{E}{A+E^2}\right)\left[1 - \frac{B-A+1}{ROOT}\right] \quad , \tag{A.29}$$

$$u_1 = \left\{\left[\frac{D^2(A-D^2)}{2Am\omega_1(A+D^2)^2} - \left(\frac{k}{4Am^2\omega_1^3}\right)\left(\frac{D}{A+D^2} + \frac{E}{A+E^2}\right)\right]\left[1 + \frac{B-A+1}{ROOT}\right]\right.$$

$$+\left(\frac{E^2(A-E^2)}{2Am\omega_1(A+E^2)^2}\right)\left[1-\frac{B-A+1}{ROOT}\right]\Bigg\} \quad , \qquad (A.30)$$

$$\dot{v}_1 = \frac{k}{4Am^2\omega_1^2}\left(\frac{D}{A+D^2}+\frac{E}{A+E^2}\right)\left[1+\frac{B-A+1}{ROOT}\right] \qquad (A.31)$$

$$\dot{u}_2 = \left\{\left[\frac{E^3}{m\omega_2(A+E^2)^2}-\left(\frac{k}{4Am^2\omega_2^3}\right)\left(\frac{D^2}{A+D^2}+\frac{E^2}{A+E^2}\right)\right]\left[1-\frac{B-A+1}{ROOT}\right]\right.$$

$$\left.+\left(\frac{D^3}{m\omega_2(A+D^2)^2}\right)\left[1+\frac{B-A+1}{ROOT}\right]\right\} \quad , \qquad (A.32)$$

and

$$\dot{v}_2 = \frac{k}{4Am^2\omega_2^2}\left(\frac{D^2}{A+D^2}+\frac{E^2}{A+E^2}\right)\left[1-\frac{B-A+1}{ROOT}\right] \qquad (A.33)$$

Substituting equations (A.26 - A.33) into equations (A.24) and (A..25) yields the exact solution of $Q_1(T)$ and $Q_2(T)$ for determining the parameter B.

APPENDIX B

a) Herein we present the closed form solutions for Q_1 and Q_2 associated with the identification of (1) parameter A, (2) parameter B and (3) parameter α from response data obtained using a sinusoidal base motion.

The solution to equation (101) when $f(t) = c_0\text{Sin}(wt)$ $(\alpha \geq 0)$ can be shown as

$$\{x(t)\} = \begin{Bmatrix} H_1\text{Sin}(\omega t) + I_1\,\text{Cos}(\omega t) \\ H_2\text{Sin}(\omega t) + I_2\text{Cos}(\omega t) \end{Bmatrix} \tag{B.1}$$

where

$$H_1 = \frac{L_1}{\sqrt{m(A+D^2)}} + \frac{L_2}{\sqrt{m(A+E^2)}} \quad , \tag{B.2}$$

$$H_2 = \frac{DL_1}{\sqrt{m(A+D^2)}} + \frac{EL_2}{\sqrt{m(A+E^2}} \quad , \tag{B.3}$$

$$I_1 = \frac{J_1}{\sqrt{m(A+D^2)}} + \frac{J_2}{\sqrt{m(A+E^2)}} \quad , \tag{B.4}$$

$$I_2 = \frac{DJ_1}{\sqrt{m(A+D^2)}} + \frac{EJ_2}{\sqrt{m(A+E^2)}} \quad , \tag{B.5}$$

and

$$L_1 = \frac{R(\lambda_1 - \omega^2)}{(\lambda_1 - \omega^2)^2 + \alpha^2\omega^2\lambda_1^2} = \frac{R\left[\left(\frac{\omega_1}{\omega_0}\right)^2 - \gamma^2\right]}{\omega_0^2\left\{\left[\left(\frac{\omega_1}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\frac{\omega_1}{\omega_0}\right)^4\right\}}, \tag{B.6}$$

$$L_2 = \frac{S\,(\lambda_2 - \omega^2)}{(\lambda_2 - \omega^2)^2 + \alpha^2\omega^2\lambda_2^2} = \frac{S\left[\left(\frac{\omega_2}{\omega_0}\right)^2 - \gamma^2\right]}{\omega_0^2\left\{\left[\left(\frac{\omega_2}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\frac{\omega_2}{\omega_0}\right)^4\right\}}, \tag{B.7}$$

$$J_1 = \frac{-R\alpha\omega\lambda_1}{(\omega^2 - \lambda_1)^2 + \alpha^2\omega^2\lambda_1^2} = \frac{-R\alpha\gamma\omega_0\left(\frac{\omega_1}{\omega_0}\right)^2}{\omega_0^2\left\{\left[\left(\frac{\omega_1}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\frac{\omega_1}{\omega_0}\right)^4\right\}}, \tag{B.8}$$

and

$$J_2 = \frac{-S\alpha\omega\lambda_2}{(\omega^2 - \lambda_2)^2 + \alpha^2\omega^2\lambda_2^2} = \frac{-S\alpha\gamma\omega_0\left(\frac{\omega_2}{\omega_0}\right)^2}{\omega_0^2\left\{\left[\left(\frac{\omega_2}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\frac{\omega_2}{\omega_0}\right)^4\right\}}, \tag{B.9}$$

with

$$\gamma = \frac{\omega}{\omega_0} , \tag{B.10}$$

$$\omega_0 = \sqrt{k/m} , \tag{B.11}$$

$$\omega^2_{1,2} \triangleq \lambda_{1,2} = \frac{k}{2Am}[(A+B+1) \pm \sqrt{(A+B+1)^2 -4AB]} , \tag{B.12}$$

$$R = \frac{-mC_0 (A+D)}{\sqrt{m(A+D^2)}} , \tag{B.13}$$

$$S = \frac{-mC_0(A+E)}{\sqrt{m(A+E^2)}} , \tag{B.14}$$

$$D = \frac{k}{k - \lambda_1 m} , \quad \text{and} \tag{B.15}$$

$$E = \frac{k}{k - \lambda_2 m} . \tag{B.16}$$

(1) If the OSL for the "best" estimation of parameter A is sought (using the approximation $\frac{\partial \omega_{d_i}}{\partial A} \approx \frac{\partial \omega_i}{\partial A}$) ; i = [1,2]), $Q_1(T)$ may be written as,

$$Q_1(T) = (\xi_1^2 + \xi_2^2)\left[\frac{T}{2} - \frac{1}{2\omega}\mathrm{Cos}\,(\omega t + 2\phi)\mathrm{Sin}(\omega t)\right] \tag{B.17}$$

where,

$$\phi = \tan^{-1}\frac{\xi_2}{\xi_1}\,. \tag{B.18}$$

The quantities ξ_1 and ξ_2 in the above equations can be expressed as:

$$\xi_1 = \left\{\frac{C_0L_1(D-D^2)}{R(A+D^2)^2} + \frac{C_0L_2(E-E^2)}{S(A+E^2)^2} + \frac{C_0D^2L_1(A-2AD-D^2)}{2RA^2(A+D^2)^2}\right.$$

$$\left[B+1+ROOT - \frac{A(A-B+1)}{ROOT}\right] + \frac{C_0E^2L_2(A-2AE-E^2)}{2SA^2(A+E^2)^2}$$

$$\left[B+1-ROOT + \frac{A(A-B+1)}{ROOT}\right] + \frac{C_0F_1k(A+D)}{2A^2m(A+D^2)}\left[\frac{A(A-B+1)}{ROOT} - (B+1)\right.$$

$$\left.\left. - ROOT + \frac{C_0F_2k(A+E)}{2A^2m(A+E^2)}\left[\frac{-A(A-B+1)}{ROOT} - (B+1) + ROOT\right]\right]\right\}, \tag{B.19}$$

and

$$\xi_2 = \left\{\frac{C_0(D-D^2)J_1}{R(A+D^2)^2} + \frac{C_0(E-E^2)J_2}{S(A+E^2)^2} + \frac{C_0D^2J_1(A-2AD-D^2)}{2RA^2(A+D^2)^2}\right.$$

$$\left[B+1+ROOT - \frac{A(A-B+1)}{ROOT}\right] + \frac{C_0E^2J_2(A - 2AE - E^2)}{2SA^2(A+E^2)^2}\left[B+1 - ROOT + \right.$$

$$\left.\frac{A(A-B+1)}{ROOT}\right] + \frac{C_0(A+D)G_1k}{2A^2m(A+D^2)}\left[\frac{A(A-B+1)}{ROOT} - (B+1+ROOT)\right]$$

$$\left. + \frac{C_0(A+E)G_2k}{2A^2m(A+E^2)}\left[\frac{-A(A-B+1)}{ROOT} - (B+1) + ROOT\right]\right\} \tag{B.20}$$

where

$$F_i = \frac{-(\alpha\omega_0)^2\gamma^2\left(\frac{\omega_i}{\omega_0}\right)^4 + \left[\left(\frac{\omega_i}{\omega_0}\right)^2 - \gamma^2\right]^2 + 2\gamma^2(\alpha\omega_0)^2\left(\frac{\omega_i}{\omega_0}\right)^2\left[\left(\frac{\omega_i}{\omega_0}\right)^2 - \gamma^2\right]}{\omega_0^4\left\{\left[\left(\frac{\omega_i}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\frac{\omega_i}{\omega_0}\right)^4\right\}^2} \tag{B.21}$$

and

$$G_i = \frac{\alpha\gamma\left\{\left[\left(\frac{\omega_i}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\frac{\omega_i}{\omega_0}\right)^4\right\}}{\omega_0^3\left\{\left[\left(\frac{\omega_i}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\frac{\omega_i}{\omega_0}\right)^4\right\}^2}$$

$$\frac{-2\left(\frac{\omega_i}{\omega_0}\right)^2\left[\left(\frac{\omega_i}{\omega_0}\right)^2-\gamma^2+(\alpha\omega_0)^2\gamma^2\left(\frac{\omega_i}{\omega_0}\right)^2\right]}{\omega_0^3\left\{\left[\left(\frac{\omega_i}{\omega_0}\right)^2-\gamma^2\right]^2+(\alpha\omega_0)^2\gamma^2\left(\frac{\omega_i}{\omega_0}\right)^4\right\}^2} \tag{B.22}$$

for i ε [1, 2].

and

$$\text{ROOT} = \sqrt{(A+B+1)^2 - 4AB} \ .$$

Similarly,

$$Q_2(T) = (\xi_1'^2 + \xi_2'^2)\left[\frac{T}{2} - \frac{1}{2\omega}\text{Cos}\,(\omega t + 2\phi')\,\text{Sin}(\omega t)\right] \tag{B.23}$$

where

$$\phi' = \tan^{-1}\frac{\xi_2'}{\xi_1'} \ ,$$

$$\xi_1' = \left\{\frac{C_0(D^2 - D^3)L_1}{R(A+D^2)^2} + \frac{C_0(E^2 - E^3)L_2}{S(A+E^2)^2} + \frac{C_0D^2L_1(A^2 - AD^2 + 2AD)}{2RA^2(A+D^2)^2}\right.$$

$$\left[B+1+ROOT - \frac{A(A-B+1)}{ROOT}\right] + \frac{C_0E^2L_2(A^2 - AE^2 + 2AE)}{2SA^2(A+E^2)^2}\left[B+1 - ROOT +\right.$$

$$\left.\frac{A(A-B+1)}{ROOT}\right] + \frac{C_0DkF_1\,(A+D)}{2A^2m(A+D^2)}\left[\frac{A(A-B+1)}{ROOT} - (B+1) - ROOT\right]$$

$$+ \frac{C_0EkF_2(A+E)}{2A^2m(A+E^2)}\left[\frac{-A(A-B+1)}{ROOT} - (B+1) + ROOT\right] \qquad (B.24)$$

and

$$\xi_1' = \left\{\frac{C_0(D^2 - D^3)J_1}{R(A+D^2)^2} + \frac{C_0(E^2 - E^3)J_2}{S(A+E^2)^2} + \frac{C_0J_1D^2(A^2 - AD^2 + 2AD)}{2RA^2(A+D^2)^2}\right.$$

$$\left[B+1+ROOT - \frac{A(A-B+1)}{ROOT}\right] + \frac{C_0J_2E^2(A^2 - AE^2 + 2AE)}{2SA^2(A+E^2)^2}\left[B+1 - ROOT +\right.$$

$$\left.\frac{A(A-B+1)}{ROOT}\right] + \frac{C_0DkG_1\,(A+D)}{2A^2m(A+D^2)}\left[\frac{A(A-B+1)}{ROOT} - (B+1) - ROOT\right]$$

$$+ \frac{C_0EkG_2(A+E)}{2A^2m(A+E^2)}\left[\frac{-A(A-B+1)}{ROOT} - (B+1) + ROOT\right] \qquad (B.25)$$

(2) If the OSL for the "best" estimation of parameter B is sought, $Q_1(T)$ may be written as

$$Q_1(T) = (v_1^2 + v_2^2)\left[\frac{T}{2} - \frac{1}{2\omega}\mathrm{Cos}\,(\omega t + 2\psi)\,\mathrm{Sin}(\omega t)\right] \tag{B.26}$$

where

$$\psi = \tan^{-1}\frac{v_2}{v_1}\ , \tag{B.27}$$

$$v_1 = \left\{\left[\frac{C_0L_1D^2(D^2 + 2AD\text{-}A)}{2RA(A+D^2)^2} + \frac{C_0F_1k(A+D)}{2Am(A+D^2)}\right]\left[1 + \frac{B\text{-}A+1}{ROOT}\right]\right.$$

$$\left. + \left[\frac{C_0L_2E^2(E^2+2AE\text{-}A)}{2SA(A+E^2)^2} + \frac{C_0F_2k(A+E)}{2Am(A+E^2)}\right]\left[1 - \frac{B\text{-}A+1}{ROOT}\right]\right\} \tag{B.28}$$

and

$$v_2 = \left\{\left[\frac{C_0J_1D^2(D^2 + 2AD\text{-}A)}{2AR(A+D^2)^2} + \frac{C_0kW_1(A+D)}{2Am(A+D^2)}\right]\left[1 + \frac{B\text{-}A+1}{ROOT}\right]\right.$$

$$\left. + \left[\frac{C_0J_2E^2(E^2+2AE\text{-}A)}{2SA(A+E^2)^2} + \frac{C_0kW_1(A+E)}{2Am(A+E^2)}\right]\left[1 - \frac{B\text{-}A+1}{ROOT}\right]\right\} \tag{B.29}$$

Similarly

$$Q_2(T) = (v_1'^2 + v_2'^2)\left[\frac{T}{2} - \frac{1}{2\omega}\mathrm{Cos}\,(\omega t + 2\psi')\,\mathrm{Sin}(\omega t)\right] \tag{B.30}$$

where

$$\psi' = \tan^{-1} \frac{\dot{\nu}_2}{\dot{\nu}_1} , \tag{B.31}$$

$$\begin{aligned}\dot{\nu}_1 = \Bigg\{ & \left[\frac{-C_0L_1D^2(A^2 - 2AD^2 + 2AD)}{2RA(A+D^2)^2} + \frac{C_0F_1kD(A+D)}{2Am(A+D^2)} \right] \left[1 + \frac{B-A+1}{ROOT} \right] + \\ & \left[\frac{-C_0L_2E^2(A^2 - 2AE^2 + 2AE)}{2SA(A+E^2)^2} + \frac{C_0F_2kE(A+E)}{2Am(A+E^2)} \right] \left[1 - \frac{B-A+1}{ROOT} \right] \Bigg\} \end{aligned} \tag{B.32}$$

and

$$\begin{aligned}\dot{\nu}_2 = \Bigg\{ & \left[\frac{-C_0J_1D^2(A^2 - 2AD^2 + 2AD)}{2AR(A+D^2)^2} + \frac{C_0W_1Dk(A+D)}{2Am(A+D^2)} \right] \left[1 + \frac{B-A+1}{ROOT} \right] + \\ & \left[\frac{-C_0J_2E^2(A^2 - 2AE^2 + 2AE)}{2SA(A+E^2)^2} + \frac{C_0W_2Ek(A+E)}{2Am(A+E^2)} \right] \left[1 - \frac{B-A+1}{ROOT} \right] \Bigg\} \end{aligned} \tag{B.33}$$

Also in equation (B.28) and (B.32)

$$W_i = \frac{\gamma(\alpha\omega_0) \left\{ \left[\left(\frac{\omega_i}{\omega_0} \right)^2 - \gamma^2 \right]^2 - (\alpha\omega_0)^2\gamma^2 \left(\frac{\omega_i}{\omega_0} \right)^4 - 2\left(\frac{\omega_i}{\omega_0} \right)^2 \left[\left(\frac{\omega_i}{\omega_0} \right)^2 - \gamma^2 \right] \right\}}{\omega_0^4 \left\{ \left[\left(\frac{\omega_i}{\omega_0} \right)^2 - \gamma^2 \right]^2 + (\alpha\omega_0)^2\gamma^2 \left(\frac{\omega_i}{\omega_0} \right)^4 \right\}^2} \tag{B.34}$$

$i \varepsilon [1, 2]$.

(3) A similar type of solution can be provided for $Q_1(T)$ and $Q_2(T)$, when OSL for "best" estimate of α is desired. Therefore one may write

$$Q_1(T) = (\mu_1^2 + v_2^2)\left[\frac{T}{2} - \frac{1}{2\omega}\text{Cos}(\omega t + 2\beta)\,\text{Sin}(\omega t)\right] \tag{B.35}$$

and

$$Q_2(T) = (\mu_1^{'2} + \mu_2^{'2})\left[\frac{T}{2} - \frac{1}{2\omega}\text{Cos}(\omega t + 2\beta')\,\text{Sin}(\omega t)\right] \tag{B.36}$$

where

$$\beta = \tan^{-1}\frac{\mu_2}{\mu_1}\quad , \tag{B.37}$$

$$\beta' = \tan^{-1}\frac{\mu_2'}{\mu_1'}\quad , \tag{B.38}$$

$$\mu_1 = \left\{\frac{2C_0(A+D)(\alpha\omega_0)\gamma^2\left(\frac{\omega_1}{\omega_0}\right)^4\left[\left(\frac{\omega_1}{\omega_0}\right)^2 - \gamma^2\right]}{\omega_0^3\left\{\left[\left(\frac{\omega_1}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\frac{\omega_1}{\omega_0}\right)^4\right\}^2 (A+D^2)}\right.$$

$$+\frac{2C_0(A+E)\,(\alpha\omega_0)\gamma^2\left(\frac{\omega_2}{\omega_0}\right)^4\left[\left(\frac{\omega_2}{\omega_0}\right)^2-\gamma^2\right]}{\omega_0^3\left\{\left[\left(\frac{\omega_2}{\omega_0}\right)^2-\gamma^2\right]^2+(\alpha\omega_0)^2\gamma^2\left(\frac{\omega_2}{\omega_0}\right)^4\right\}^2(A+E^2)}\Bigg\}, \tag{B.39}$$

$$\mu_2=\Bigg\{\frac{C_0\gamma(A+D)\left(\frac{\omega_1}{\omega_0}\right)^2}{\omega_0^3(A+D^2)\left\{\left[\left(\frac{\omega_1}{\omega_0}\right)^2-\gamma^2\right]^2+(\alpha\omega_0)^2\gamma^2\left(\frac{\omega_1}{\omega_0}\right)^4\right\}}$$

$$-\frac{2C_0(A+D)(\alpha\omega_0)^2\gamma^3\left(\frac{\omega_1}{\omega_0}\right)^6}{\omega_0(A+D^2)\left\{\left[\left(\frac{\omega_1}{\omega_0}\right)^2-\gamma^2\right]^2+(\alpha\omega_0)^2\gamma^2\left(\frac{\omega_1}{\omega_0}\right)^4\right\}^2}$$

$$+\frac{C_0\gamma(A+E)\left(\frac{\omega_2}{\omega_0}\right)^2}{\omega_0^3(A+E^2)\left\{\left[\left(\frac{\omega_2}{\omega_0}\right)^2-\gamma^2\right]^2+(\alpha\omega_0)^2\gamma^2\left(\frac{\omega_2}{\omega_0}\right)^4\right\}}$$

$$- \frac{2C_0(A+E)(\alpha\omega_0)^2\gamma^3\left(\frac{\omega_2}{\omega_0}\right)^6}{\omega_0(A+E^2)\left\{\left[\left(\frac{\omega_1}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\frac{\omega_2}{\omega_0}\right)^4\right\}^2} \tag{B.40}$$

$$\mu_1' = \left\{ \frac{C_0D(A+D)(\alpha\omega_0)\gamma^2\left(\frac{\omega_1}{\omega_0}\right)^4\left[\left(\frac{\omega_1}{\omega_0}\right)^2 - \gamma^2\right]}{\omega_0^3(A+D^2)\left\{\left[\left(\frac{\omega_1}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\frac{\omega_1}{\omega_0}\right)^4\right\}^2} \right.$$

$$+ \frac{2C_0E(A+E)\,(\alpha\omega_0)\gamma^2\left(\frac{\omega_2}{\omega_0}\right)^4\left[\left(\frac{\omega_2}{\omega_0}\right)^2 - \gamma^2\right]}{\omega_0^3(A+E^2)\left\{\left[\left(\frac{\omega_2}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\frac{\omega_2}{\omega_0}\right)^4\right\}^2} \tag{B.41}$$

and

$$
\mu_2' = \left\{ \frac{C_0 D\gamma(A+D)\left(\dfrac{\omega_1}{\omega_0}\right)^2}{\omega_0^3(A+D^2)\left\{\left[\left(\dfrac{\omega_1}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\dfrac{\omega_1}{\omega_0}\right)^4\right\}} \right.
$$

$$
- \frac{2C_0 D\gamma^3(A+D)(\alpha\omega_0)^2\left(\dfrac{\omega_1}{\omega_0}\right)^6}{\omega_0(A+D^2)\left\{\left[\left(\dfrac{\omega_1}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\dfrac{\omega_1}{\omega_0}\right)^4\right\}^2}
$$

$$
+ \frac{C_0 E\gamma(A+E)\left(\dfrac{\omega_2}{\omega_0}\right)^2}{\omega_0^3(A+E^2)\left\{\left[\left(\dfrac{\omega_2}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\dfrac{\omega_2}{\omega_0}\right)^4\right\}}
$$

$$
- \frac{2C_0 E\gamma^3(A+E)(\alpha\omega_0)^2\left(\dfrac{\omega_2}{\omega_0}\right)^6}{\omega_0(A+E^2)\left\{\left[\left(\dfrac{\omega_1}{\omega_0}\right)^2 - \gamma^2\right]^2 + (\alpha\omega_0)^2\gamma^2\left(\dfrac{\omega_2}{\omega_0}\right)^4\right\}^2} \qquad \text{(B.42)}
$$

(b) An easier approach to finding the OSL for the response measurement to identify A or B can be used for the case $\alpha = 0$. Writing the response $X = Y\mathrm{Sin}(\omega t)$ we have,

$$Y = [\Omega]^{-1} \begin{Bmatrix} -Am \\ -m \end{Bmatrix} \tag{B.43}$$

where

$$[\Omega] = \begin{bmatrix} (B+1)k - Am\omega^2 & -k \\ -k & k - m\omega^2 \end{bmatrix} \tag{B.44}$$

Hence we have, after some simplification,

$$\begin{Bmatrix} \dfrac{\partial Y}{\partial A} \end{Bmatrix} = \frac{-1}{|\Omega|^2} \begin{Bmatrix} Bkm\,(k-m\omega^2)^2 \\ Bk^2m\,(k-m\omega^2) \end{Bmatrix} , \tag{B.45}$$

where

$$|\,\Omega\,| = (k-m\omega^2)\,[k(B+1) - Am\omega^2] - k^2 \ . \tag{B.46}$$

This yields

$$Q_1(T) = P_0 \left[\frac{Bkm(k-m\omega^2)^2}{|\Omega|^2} \right]^2 \tag{B.47}$$

$$Q_2(T) = P_0 \left[\frac{Bk^2 m(k-m\omega^2)}{|\Omega|^2} \right]^2 \tag{B.48}$$

where

$$P_0 = \frac{T}{2} - \frac{\text{Sin}\,(2\omega T)}{4\omega}. \tag{B.49}$$

Thus

$$\frac{Q_1(T)}{Q_2(T)} = (1 - \gamma^2)^2 , \quad \gamma \neq 1 \tag{B.50}$$

where

$$\gamma = \omega\left(\frac{k}{m}\right)^{-1/2} . \tag{B.51}$$

We observe that though the ratio $Q_1(T)/Q_2(T)$ tends to zero as γ tends to unity, indicating that the upper story is the prefered sensor location, the values of $Q_1(T)$ and $Q_2(T)$ becomes vanishingly small for $\gamma = 1$. Also $Q_1(T) = Q_2(T)$ for $\gamma = 0$ and $\gamma = \sqrt{2}$.

Similarly, the expressions for $Q_1(T)$ and $Q_2(T)$ associated with the identification of B can be expressed as:

$$Q_1(T) = P_0 \left[\frac{km(k-m\omega^2)\,[A(k-m\omega^2) + k]}{|\Omega|^2} \right]^2 , \tag{B.52}$$

and

$$Q_2(T) = P_0\left[\frac{k^2m[A(k-m\omega^2) + k]}{|\Omega|^2}\right]^2 , \tag{B.53}$$

with the ratio given by

$$\frac{Q_1(T)}{Q_2(T)} = (1 - \gamma^2)^2, \qquad \gamma \neq \sqrt{1 + \frac{1}{A}} . \tag{B.54}$$

For $\gamma = \sqrt{1 + \frac{1}{A}}$, the ratio $\frac{Q_1}{Q_2} = \frac{1}{A^2}$; however equations (B.52 - B.53) indicate

that for $\gamma = \sqrt{1 + \frac{1}{A}}$ both $Q_1(T)$ and $Q_2(T)$ become vanishingly small. Once again, $Q_1(T) = Q_2(T)$ for $\gamma = 0$, and $\gamma = \sqrt{2}$, $(A \neq 1)$.

APPENDIX C

Taking the first variation of equation (235) we obtain

$$\delta J = -\alpha \int_0^T \{\delta y^T Q\, y\} dt + \beta \int_0^T \{\delta y^T H^T R^{-1} H y\} dt - \lambda^T(t)\, \delta y \,\Big|_0^T$$

$$+ \int_0^T \{ \dot{\lambda}^T + \lambda^T F\}\, \delta y\, dt + \int_0^T \lambda^T G\, \delta f\, dt + \int_0^T \eta(t)\, \delta f^T f\, dt - \frac{\eta(t)}{2}\, \delta y_{N+1} \,\Big|_0^T$$

$$+ \int_0^T \frac{\dot{\eta}}{2}\, \delta y_{N+1}\, dt \qquad \text{(C.1)}$$

Collecting terms in δy, δf and $\delta \eta$, we get the relations:

$$\dot{\lambda}(t) + F^T \lambda(t) = -\alpha \{Q\} y + \beta \{H^T R^{-1} H\} y, \qquad \text{(C.2)}$$

$$f(t) = -\frac{G^T \lambda(t)}{\eta(t)} \qquad \text{(C.3)}$$

$$\dot{y}_{N+1}(t) = f^T f = \frac{\lambda^T(t)\, G\, G^T \lambda(t)}{\eta(t)^2} \qquad \text{(C.4)}$$

$$\dot{\eta} = 0 \qquad \text{(C.5)}$$

along with the boundary conditions

$$y_{N+1}(0) = 0 \; ; \quad y_{N+1}(T) = E \; ; \quad \lambda(T) = 0 \tag{C.6}$$

Hence the result given by equation (236).